UNIVERSITY OF MILAN-BICOCCA
Master's Thesis in Theoretical Physics

Thermodynamic Aspects of AdS_4 Black Holes in $\mathcal{N} = 2$ Gauged Supergravity

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The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time.

S. Chandrasekhar, *"The Mathematical Theory of Black Holes"*

Abstract

The objective of this work is to analyze the relation between the on-shell gravitational action and the entropy for a class of *AdS* black holes, following the lines of a recent work by Halmagyi and Lal, [1]. The relation states that the on-shell gravitational action, when evaluated at its BPS limit, equals minus the black hole entropy, calculated as the area of the event horizon.

The analysis is done for static, magnetically charged black holes in AdS_4 arising as solutions in $\mathcal{N} = 2$ Fayet-Iliopoulos gauged supergravity coupled with running scalars. In particular, we are investigating the validity of the above relation, as stated in [1] for generic AdS_4 black holes, for the solutions in another recent work, by Gnecchi and Toldo, [2], which treats the particular class considered here. The comparison exhibits an apparent contraddiction, as the results in [2] seem not to satisfy the action-entropy relation.

Contents

Introduction

Black holes are regions of spacetime where the gravitational force is so strong that not even light can escape. The event horizon is the boundary to the region of no return, shielding the singularity at the center of the black hole, where the curvature becomes infinite. They arise as solution to the Einstein equation

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor. $G_{\mu\nu}$ contains information about the metric and so the curvature of spacetime, whereas $T_{\mu\nu}$ describes the matter distribution that causes the curvature.
Studying black holes in General Relativity reveals interesting results. First of all they are macroscopic object, as to completely describe them three quantities are sufficient. They are referred to as charges and are mass, angular momentum, magnetic and electric charge. One can imagine that when a black hole forms from gravitational collapse then the creation of an event horizon has the effect of smoothing out all the particularities of the starting situation leaving us with the only three quantities above. This is referred to as *no-hair theorem* and the idea that all the information about the internal structure has disappeared behind the horizon is commonly expressed with the phrase *black holes have no hair*.

On the other hand black holes are thermodynamic objects with all their thermodynamic properties depending on the event horizon only and not on the interior. In seminal works by Bekenstein ([3], [4], [5]) and Hawking ([6], [7], [8]), in fact, a remarkable correspondence between the laws of black hole mechanics and the laws of thermodynamics was established. Hawking showed that black holes are not so black as they emit radiation and thus have a temperature. The radiation is thermal meaning that it provides no information about the black hole interior leading to the *information loss paradox* as it is in contrast with the unitary evolution of any quantum system and still remains an open problem.

The major question that we address here is the fact that a black hole possesses an entropy which is equal to one quarter of its event horizon area, a result known as *Bekenstein-Hawking area law*. According to Boltzmann, thermodynamic entropy has a statistical interpretation in terms of microscopic configurations that give rise to the same macroscopic properties. Identifying the black hole microstates is a difficult task, since, due to no-hair theorem we have no access to any information on its internal configuration. A key issue in theoretical physics is to give a consistent interpretation of the black hole microstates and reproduce the Bekenstein-Hawking entropy by directly counting them.

Since regimes of very strong gravity are involved, a microscopic explanation of the Bekenstein-Hawking entropy can only be found in a theory of quantum gravity, a theory that unifies gravitation and quantum mechanics. String theory is the most successful candidate. The basic assumption is that the most fundamental constituents are not point-like, but are one-dimensional strings, that can be either open or closed. String theory also predicts the existence of D-branes which are extended objects open strings can end on. Stack of D-branes can reproduce the geometry of a black hole and for this reason the first attempt to find a microscopic explanation for black hole entropy, made by Strominger and Vafa in 1996, [9], related the black hole microstates with the counting of configurations of D-branes. The analysis was done for asymptotically flat balck holes in five dimensions.

Subsequent development of the AdS/CFT correspondence in 1997 by Maldacena, [10], a duality relation between a gravitational theory in a d-dimensional AdS spacetime and a conformal field theory (CFT) in $(d-1)$-dimensions, has brought interest in asymptotically anti-de Sitter black holes, since the correspondence would allow to perform calculations in the dual field theory. The correspondence is a realization of the holographic principle, stating that the description of a volume of spacetime is encoded on the boundary of that volume. Black hole entropy is indeed a holographic quantity since it depends on the area (and not the volume) of the black hole horizon. AdS black hole microstates are realized as degrees of freedom of the corresponding field theory, by means of the field theory partition function.

The black holes that are studied in this framework are supersymmetric black holes (called BPS black holes) which arise as solutions to a supergravity theory (a low energy effective theory for string theory) with a negative cosmological constant, usually coupled to scalar fields. The analysis of more realistic black holes remains an unsolved issue.

Objective and Main Results

The microscopic explanation of the entropy for AdS black holes employing the correspondence, however, has not been developed until very recently, ([11]). The microscopic counting is done on the field theory side, where a quantity known as the supersymmetric index is computed, ([12]). The index represents the supersym-

metric field theory partition function. In the large N limit, the entropy of AdS black holes can be extracted from the index. Since the computation involves extremizing the index with respect to a set of chemical potentials, this is known as the extremization principle (see [13]-[15]).
On the gravitational side, instead, we have that, thanks to holography, the logarithm of the supersymmetric partition function of the CFT reproduces the on-shell action. As a consistency check we should have that the on-shell action is related to the entropy in the supersymmetric limit, (see [1], [16], [17]). In particular, the on-shell action for BPS black holes in AdS should equal minus the black hole entropy, calculated as the area of the event horizon.

$$S_{\text{on-shell}}\Big|_{BPS} = -S \tag{1}$$

For the case in exam, the duality is between a four-dimensional supergravity theory, which admits asymptotically AdS_4 black hole solutions, and a three-dimensional supersymmetric CFT living on the conformal boundary, which is the $ABJM$ theory. In this work we are considering the gravitational side only to explore the relation between the on-shell gravitational action and the black hole entropy, in the BPS limit. In particular we are comparing two recent works, one by Halmagyi and Lal, [1], and the other by Gnecchi and Toldo, [2], which exhibit an apparent contradiction when it comes to check the consistency between gravitational action and entropy.

The computation of the gravitational action is highly non trivial and necessitates to be done in the non-extremal case (i.e. for black holes with non-zero temperature and thus non supersymmetric). At the end of calculations the limit to supersymmetric solution should be performed. Along these lines, an explicit example of the validity of (1) can be found for the so-called universal black hole[1] in [16]. Here the calculations were done for a non-extremal deformation of the solution and then the limit to BPS configuration was performed. The matching between the on-shell gravitational action and the entropy has also been recently discussed in [1], to which we refer in this work, and in [17].

In the following work we address the issue (1) for static, magnetically charged black holes with spherical horizon in AdS_4, resulting from $\mathcal{N} = 2$ four-dimensional FI (Fayet-Iliopoulos) gauged supergravity (a supergravity theory is a low energy effective theory for string theory), coupled to one running scalar field. We are considering $1/4$ BPS black holes, preserving two out of 8 supercharges. The first $1/4$ BPS black hole solution in AdS_4 with radially dependent scalar profiles were found by Cacciatori and Klemm ([20]) and are the starting point for the magnetic solution

[1]This the supersymmetric magnetically charged AdS_4 black hole ([18], [19]) with constant scalar profiles

proposed in [2], which we will follow closely.

- In Gnecchi and Toldo, [2] the class of static, magnetically charged AdS_4 black holes, objective of this work, is analyzed and it results that magnetic black holes are a state in a canonical thermodynamic ensemble, characterized by Helmholtz free energy, $\Omega = M - TS$. Holography relates the on-shell gravitational action to the free energy,

$$S_{\text{on-shell}} = \beta\Omega = \beta(M - TS)$$

  with $\beta = 1/T$. However, the BPS limit of the on-shell action does not reproduce the black hole entropy as the black hole mass, computed by means of holographic renormalization techniques, would not vanish when taking the limit. We are expecting the black hole mass to be zero in the BPS limit, since, it is the only way to get rid of the divergence as $T \to 0$ in $\beta(M - TS)$ and satisfy (1).
  This also happens to be the condition to saturate the BPS bound for magnetic black holes, as established in [21] and [22]

$$M \geq 0$$

- The work by Halmagyi and Lal, [1], instead, states that (1) holds for generic dyonic black holes in AdS_4, by directly computing the on-shell gravitational action.

The discrepancy is due to presence of finite terms coming from the delicate choice of boundary conditions for the scalar fields, ([23], [24]), as well as of the counterterms used to regularize the divergencies. We obtain that the choice of Neumann boundary conditions and counterterms, as is done in [1], leads to the right mass for (1) to hold. The mass calculated via holographic renormalization and using Neumann boundary conditions, as well, saturates the BPS limit for magnetic black holes.
The choice in [2] (mixed boundary conditions), instead, would lead to a non-vanishing mass in the extremal limit which is the reason why the relation in (1) fails in this case.

The subtle choice between Neumann and mixed boundary conditions reflects when checking the first law of thermodynamics. It seems that the mass which satisfy the first law is not the one saturating the BPS bound.

Thesis Outline

In the following we outline the content of single chapters

- In *chapter 1* we introduce the laws of black hole mechanics and show their correspondence with the laws of thermodynamics. In particular we recover some useful formulas, such as the relation between temperature and the warp factors appearing in the metric, which will be useful later. We do not provide a rigorous proof of the laws, but in some cases, sketch their derivation.

- In *chapter 2* the $N = 2$ supergravity theory in 4 dimensions is introduced. It is the framework for the black hole solutions that will be considered in the rest of the work. In particular, we describe the setup for the abelian gauging procedure which guarantees an AdS vacuum (Fayet-Iliopoulos gauging). We set the fermionic part of the lagrangian to zero and analyze the bosonic part only. The extremal black hole configuration and its non-extremal deformation are considered for a static, spherically symmetric and with magnetic charges only ansatz. At last we analyze the asymptotic behavior of the scalar field, which accounts for the right sign of the cosmological constant. In particular we relate the scalar field mass to possible boundary conditions that can be imposed on the field on the asymptotic boundary.

- In *chapter 3* we analyze the holographic renormalization technique to compute mass in AdS spacetime. This is a divergent quantity that exhibit linear and cubic divergences coming from the Gibbons-Hawking boundary term and they have to be removed in order to recover a finite value. Holographic renormalization evaluates mass from the tt component of the boundary stress-energy tensor, which is quantity depending on the variation of the boundary terms in the action. The variation is explicitly performed. We separately analyze the regular, finite and divergent terms, along the work in [2].

- In *chapter 4* we analyze the validity of the relation (1) for the entropy of magnetic black holes. We explore the holographic relation between the free energy of the system and its on-shell action. We directly compute the on-shell action along the lines of [1] and analyze the contradiction with the results in [2]. Then we tackle the issue of boundary conditions for the scalar field in relation to both the extremization principle and the validity of the first law of thermodynamics.

- In *chapter 5* we draw the conclusions of our work trying to resolve the apparent inconsistency between [2] and [1]. An open question remains on the mass satisfying the first thermodynamic law and the one satisfying the BPS bound.

1 Black Hole Thermodynamics

There is a remarkable correspondence between the laws of black hole mechanics and the laws of thermodynamics, [8]. As long as our treatment remains classical the resemblance is just an analogy. If we allow for a quantum mechanical analysis, then it turns out that black holes can radiate, [6], possess a temperature and an entropy. As a consequence, they can really be thought of as true thermodynamic systems. In particular, we will see that both temperature and entropy depend only on the event horizon. Temperature is related to the surface gravity on the horizon and the entropy is equal to the horizon area.
Here we give a brief review of the laws. Some original works were referred to in the Introduction and one can also see the dedicated chapters in [25] and [26], the review given in [27] or the lectures notes given in [28] and [29].

1.1 The Zeroth Law

The zeroth law of black hole mechanics states that a quantity called surface gravity, usually indicated with κ, is constant on the event horizon. This is analogue to the first law of thermodynamics which states that temperature is constant at thermal equilibrium.
Here we only state the zeroth law of black hole mechanics, without proving it. For a rigorous proof, one can refer, for example, to [30].

In what follows we are going to analyze what is κ and how it is defined. The surface gravity is so called from the fact that for a non rotating black hole, it is related to the acceleration a static observer near the horizon needs to have, as seen by a static observer at infinity, in order not to fall inside the black hole.

To define surface gravity we need the concepts of null hypersurface and Killing horizon. A hypersurface can be defined as a level set of a function. Since we are working in a 4-dimensional spacetime, it is a 3-dimensional submanifold. We call it Σ.

$$\Sigma \colon f(x^\mu) = const \tag{1.1}$$

Then we can define its normal vector as

$$df = \partial_\mu f dx^\mu \tag{1.2}$$

and its corresponding dual

$$l^\mu = g^{\mu\nu}\partial_\nu f \tag{1.3}$$

The hypersurface Σ is said to be null when l^μ is a null vector, i.e.

$$l^2\Big|_\Sigma = l \cdot l\Big|_\Sigma = l^\mu g_{\mu\nu} l^\nu\Big|_\Sigma = 0 \tag{1.4}$$

Then l^μ is both orthogonal and tangent. In fact, a null vector is normal to itself.

Using the torsion-free property of the Levi-Civita connection ($\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$) it is easy to get to the following property

$$\begin{aligned} l^\mu \nabla_\mu l^\nu &= l^\mu g^{\nu\rho} \nabla_\mu \partial_\rho f = \\ &= l^\mu g^{\nu\rho} \nabla_\rho \partial_\mu f = \\ &= \frac{1}{2} g^{\nu\rho} \nabla_\rho (l^2) \end{aligned} \tag{1.5}$$

Since l^2 is null on Σ, its gradient is normal and therefore proportional to l^μ

$$\nabla_l l^\mu \equiv l^\mu \nabla_\mu l^\nu \propto l^\nu \tag{1.6}$$

This is the defining equation for surface gravity when Σ is a Killing horizon, i.e. a hypersurface that has a null Killing vector field ξ^μ. Since we cannot have two null vectors orthogonal to each other as the metric would be degenerate, we have that ξ^μ has to be proportional to l^μ and we can write

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu \tag{1.7}$$

Surface gravity arises as the proportionality constant, κ.

For a black hole with spherical (like Schwartzschild) or axisymmetric (like Kerr) the event horizon is always a Killing horizon, as long as it is a null hypersurface. Since the event horizon is a surface of constant r, then ∂_r is normal to it. Imposing that it is a null vector results in

$$\partial_\mu r \partial_\nu r g^{\mu\nu}\Big|_{hor} = g^{rr}\Big|_{hor} = 0 \tag{1.8}$$

But this is nothing else but the condition to find the event horizon given a metric. We can therefore conclude that event horizons are Killing horizons, but the converse is not true.

1.1.1 Relation between temperature and euclidean time

As mentioned above the surface gravity κ is related to the black hole temperature thus accounting for the equivalence between the zeroth law of black hole mechanics and the corresponding one in thermodynamics.

There is a standard procedure to calculate the black hole temperature by means of Euclidean gravity. One has to perform Euclidean continuation of the near horizon geometry and then impose that the metric has the right periodicity, free of conical singularities.
A key point is that a thermal system with temperature $T = 1/\beta$ is periodic in euclidean time with period β. This can be understood when taking into consideration that the thermodynamic partition function is given by

$$Z = \mathrm{Tr} e^{-\beta H} \tag{1.9}$$

with H the Hamiltonian of the system. In quantum mechanics the evolution operator is given by e^{-iHt}. The trace imposes periodicity with period β in the Euclidean time.

We now show the relation between T and the metric warp factors. Given a metric of general form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + g(r)d\Omega_2^2 \tag{1.10}$$

Near the horizon r_h we can do the following expansion

$$\begin{aligned} f(r) &= f(r_h) + f'(r_h)(r - r_h) + ... \\ &\sim f'(r_h)(r - r_h) \end{aligned} \tag{1.11}$$

since $f(r)$ is zero when evaluated on the horizon by definition.

Concentrating on the $t - r$ coordinates and changing the radial coordinate as

$$r = r_h \left(1 + \frac{|f'(r_h)|}{4r_h}\rho^2\right) \qquad \Longrightarrow \qquad dr = \frac{1}{2}|f'(r_h)|\rho d\rho \tag{1.12}$$

we get

$$\begin{aligned} ds^2 &= \frac{\left(f'(r_h)\right)^2}{4}\rho^2 dt^2 + d\rho^2 + ... \\ &= -\rho^2 \left(\frac{|f'(r_h)|}{2}dt\right)^2 + d\rho^2 + ... \\ &= \rho^2 \left(\frac{|f'(r_h)|}{2}d\tau\right)^2 + d\rho^2 + ... \end{aligned} \tag{1.13}$$

where in the last equality we have passed to the Euclidean time $t \to -i\tau$. Near the horizon the metric resembles just the flat metric in polar coordinates. We should

stress the fact that the similarity holds only in the vicinity of the horizon.
Imposing the periodicity of imaginary time to be β leads to

$$\beta = \frac{4\pi}{|f'(r_h)|} \tag{1.14}$$

Then we can conclude that black hole temperature can be calculated directly from the metric warp functions as follows

$$T = \frac{1}{4\pi} |f(r)|' \Big|_{r_h} \tag{1.15}$$

Relation between temperature and surface gravity

On the other hand there is a relation between the surface gravity and temperature which states that

$$T = \frac{\kappa}{2\pi} \tag{1.16}$$

The above relation has been explained by Hawking in [6], where he showed that considering quantum effects near the black hole horizon causes emission and production of particles by the black hole as if it were a thermal body with a temperature given by

$$T = \frac{\hbar\kappa}{2\pi k_B} \tag{1.17}$$

where $\hbar$ is the reduced Planck constant, κ is the surface gravity and k_B is the Boltzmann constant. Setting $\hbar = k_B = 1$ we retrieve what stated in (1.16).
Form (1.15) and (1.16), it is straightforward to conclude that also the following relation holds

$$\kappa = \frac{1}{2} |f(r)|' \Big|_{r_h} \tag{1.18}$$

An example: temperature of the Reissner-Nördstrom black hole

We sketch the derivation of the temperature for a charged black hole, known as the Reissner-Nördstrom black hole in asymptotically flat spacetime. Its line element can be written as

$$ds^2_{\mathrm{RN}} = -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)} dr^2 + r^2 d\Omega_2^2 \tag{1.19}$$

where M is the mass, Q is the generic charge and $d\Omega_2^2$ the infinitesimal volume element of a 2-sphere. The horizon is defined by

$$1 - \frac{2GM}{r} + \frac{Q^2}{r^2} = 0 \tag{1.20}$$

which leads to two solutions, the outer, r_+, and the inner, r_-, horizons

$$r_\pm = GM \pm \sqrt{G^2M^2 - Q^2G} \tag{1.21}$$

r_+ is usually referred to as the horizon.

Following the procedure outlined in the previous paragraph we pass to the Euclidean time, define the new variable

$$r = r_+ \left(1 + \rho^2\right) \tag{1.22}$$

and expand about $\rho = 0$.
We get

$$ds^2 = \frac{4r_+^3}{r_+ - r_-}\left[d\rho^2 + \rho^2 \left(\frac{r_+ - r_-}{2r_+^2}\right)^2 d\tau^2 + \frac{r_+ - r_-}{4r_+} d\Omega_2^2\right] \tag{1.23}$$

It is immediate to read the value for β

$$\beta = \frac{4\pi r_+^2}{r_+ - r_-} \implies T = \frac{r_+ - r_-}{4\pi r_+^2} \tag{1.24}$$

A few comments are in order. First we observe that T is always positive as $r_+ > r_-$ and it vanishes when the two horizons coincide. Although the calculation is given for the Reissner-Nördstrom black the dipendence of temperature on the differene between the outer and the inner horizons is a general feature.
This is the extremality condition which will be addressed in chapter 2. Extremal black holes are, in fact, defined by their zero temperature, which is equivalent to requiring that $r_+ = r_-$. The tt component in metric, therefore, has to be a perfect square thus to have two coincident solutions in equation (1.20). This condition is analogue to the saturation of the bound to have real solutions (imposing the existence of the square root)

$$M \geq \frac{|Q|}{\sqrt{G}} \tag{1.25}$$

We will come back to the issue of a bound for black hole mass in chapter 4 as it will play a fundamental role for black hole in AdS spacetime analized in this project.

1.2 The First Law

The first law is concerned with mass, angular momentum and charge variation for a black hole after we throw in it a particle characterized by its particular mass, angular momentum and charge. Quantities such as mass and angular momentum are not straightforwardly defined in General Relativity. We address the mass issue in chapter 3. The first law then is expressed, in the most general case (including rotation and presence of charge), as

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ + \chi dP \tag{1.26}$$

where M is the black hole mass, κ is the surface gravity as defined in (1.7), A is the area of the event horizon, Ω is the black hole angular velocity, J its angular momentum, Φ and χ are respectively the electrostatic and magnetostatic potentials evaluated between the horizon and infinity, and Q and P are respectively the electric and magnetic charges.

To give a more consistent interpretation of the quantities above and relate (1.26) to first law of thermodinamics which states

$$dE = TdS - pdV \tag{1.27}$$

we will follow closely [3] and consider a generic black hole solution characterized by mass M, charge Q and angular momentum per unit mass a, known as Kerr-Newman solution.

$$ds^2_{\text{KN}} = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left(\left(r^2+a^2\right)d\phi - adt\right)^2 \tag{1.28}$$

where

$$\Delta = r^2 - 2GMr + a^2 + Q^2, \quad \Sigma = r^2 + a^2\cos^2\theta, \quad a = \frac{J}{M} \tag{1.29}$$

Using the previously cited result that entropy equals the area of the event horizon, a question that will be analyzed in the subsequent paragraph, we are going to differentiate the area.

First we remember that the area of the horizon is given by

$$A = \int_{r=r_+} \sqrt{g_{\theta\theta}g_{\phi\phi}}d\theta d\phi \tag{1.30}$$

For the Kerr-Newman black hole we get

$$\begin{aligned} A &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{\Sigma\left(\frac{\Delta}{\Sigma}a^2\sin^2\theta + \frac{\sin^2\theta}{\Sigma}\left(r^2+a^2\right)^2\right)}\Bigg|_{r_+} = \\ &= 2\pi\int_0^{\pi}\sin\theta\left(r_+^2 + a^2\right) = \\ &= 4\pi\left(r_+^2+a^2\right) \end{aligned} \tag{1.31}$$

where we have used that $\Delta = 0$ is the horizon equation and r_+ is the outer horizon. The two horizons are given by

$$r_\pm = GM \pm \sqrt{G^2M^2 - a^2 - Q^2} \tag{1.32}$$

Considering the rationalized area $\alpha = A/4\pi$ in order not to have π factors, we can write

$$\alpha = r_+^2 + a^2 = 2GMr_+ - Q^2 \tag{1.33}$$

where we have used that

$$\begin{aligned} \Delta|_{r_+} &= 0 \\ r_+^2 - 2GMr_+ + a^2 + Q^2 &= 0 \\ r_+^2 + a^2 &= 2GMr_+ - Q^2 \end{aligned} \tag{1.34}$$

Differentiating (1.33) we obtain

$$d\alpha = 2r_+dM + 2Mdr_+ - 2QdQ \tag{1.35}$$

From (1.32) we have

$$dr_+ = dM + \frac{MdM - ada - QdQ}{\sqrt{M^2 - a^2 - Q^2}} \tag{1.36}$$

If we solve for dM we get

$$dM = \Theta d\alpha + \vec{\Omega} d\vec{J} + \Phi dQ \tag{1.37}$$

with

$$\Theta = \frac{\frac{1}{4}(r_+ - r_-)}{\alpha}, \qquad \vec{\Omega} \equiv \frac{\vec{a}}{\alpha}, \qquad \Phi \equiv \frac{Qr_+}{\alpha} \tag{1.38}$$

We observe that Θ can be interpreted as the black hole temperature, $\vec{\Omega}$ can be identified with its angular velocity and Φ with its electric potential. In (1.37) the correspondence with (1.27) is manifest. For more details see [27] or [29]. The $\vec{\Omega}d\vec{J}$ and ΦdQ terms are the analog of the work term $-pdV$ done on a thermodynamic system. $d\alpha$ resembles the entropy and dM is the relativistic analogue of energy variation dE appearing in (1.27).
Therefore, from (1.27), we can write

$$S = \frac{Ac^3}{4\hbar G} = \frac{A}{4} \tag{1.39}$$

where in the last expressions we have set all the fundamental constants to 1.

1.3 The Second Law: Bekenstein-Hawking area theorem

The first law (1.26) relates the changes in mass, charge and angular momentum that a black hole can undergo. These quantities, however, cannot be modified arbitrarily. There is a restriction and this restriction states that the area A of a black hole cannot decrease. It remains the same when the transformation is reversible or increases when it is irreversible. This is in striking analogy with the second law of thermodynamics and it constitutes the second law of black hole mechanics.

$$\delta A \geq 0 \tag{1.40}$$

The proof of the area theorem is due to Hawking, [7]. In the following we show that it holds for the Penrose process.

We have already stressed that the event horizon of a black hole is region of no return, nothing can escape. It came out, however, thanks to the works by Penrose (see, for example [31]), that energy can be extracted from a black hole that has an ergosphere. Therefore we are referring to the Kerr black hole in the following discussion. Here we report the metric

$$ds_{\mathrm{K}}^2 = -dt^2 + \frac{2GMr}{\rho^2}\left(dt - a\sin^2 d\phi\right)^2 + \left(r^2 + a^2\right)\sin^2\theta d\phi^2 + \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) \tag{1.41}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2GMr + a^2 \tag{1.42}$$

A Kerr black holes has two Killing vectors, ∂_t and ∂_ϕ. The ergosphere is the region where the time translation Killing vector field, $\xi^\mu\partial_\mu = \partial_t$, becomes spacelike and it is a combination of ∂_t and ∂_ϕ that is timelike. This means that inside the ergosphere we are forced to follow a mix of time and ϕ directions, we are forced to rotate.

We can consider a test particle of momentum p^μ. Its energy [1] is given by

$$E = -p^\mu\xi_\mu \tag{1.43}$$

Note that in the ergosphere we have no control on the sign and this the reason why the Penrose process works. One can, in fact, arrange a situation in which a particle with total energy E_0

$$E_0 = -p_0^\mu\xi_\mu \tag{1.44}$$

falls freely towards a black hole. Since it is a free fall E_0 remains constant. Then it breaks in two pieces, with momenta p_1^μ and p_2^μ, when entering the ergosphere,

[1] Note that in this chapter we are using the mostly plus signature, then the minus sign in defining the energy assures that it is a positive quantity

in such a way that one enters the black hole and the other escapes. Furthermore suppose taht the one falling inside the black hole has negative energy, be it $E_1 < 0$. Conservation of momentum requires

$$p_0^\mu = p_1^\mu + p_2^\mu \tag{1.45}$$

Contraction with ξ_μ gives the equation for the conservation of total energy

$$E_0 = E_1 + E_2 \tag{1.46}$$

As we have arranged for E_1 to be negative, it means that E_2 is positive and, given the conservation of energy

$$E_2 = E_0 + |E_1| \tag{1.47}$$

The increase in energy by $|E_1|$ comes from the reduction of the black hole mass by the same amount. The energy $|E_1|$ has been extracted from the black hole.

To see the relation to the second law we should consider that energy extraction has a limit, imposed by the fact that the particle entering the black hole possesses negative angular momentum, which means that it rotates in the opposite direction with respect to that of the black hole. The consequence is that the black hole angular momentum can be reduced to zero, while maintaining its mass M finite. When the angular momentum vanishes there is no ergosphere and no further energy could be extracted.
The linear combination given by

$$\chi_\mu = \partial_t + \frac{a}{r_+^2 + a^2}\partial_\phi = \partial_t + \Omega\partial_\phi \tag{1.48}$$

is null on the horizon r_+, so we have that for the particle entering the back hole that

$$\begin{aligned} p_1^\mu \chi_\mu &\leq 0 \\ p_1^\mu\left(\partial_t + \Omega\partial_\phi\right) &\leq 0 \\ -E_1 + \Omega J_1 &\leq 0 \\ J_1 &\leq \frac{E_1}{\Omega} \end{aligned} \tag{1.49}$$

This justifies the statement that the angular momentum of the entering particle should be negative.

After the particle is absorbed by the black hole, its mass and angular momentum change by exactly the mass and angular momentum of the swallowed particle. So we can write

$$\delta J \leq \frac{\delta M}{\Omega} \tag{1.50}$$

and in turn it can be rewritten as

$$\delta M_{\mathrm{irr}} \geq 0 \tag{1.51}$$

where M_{irr} is the irreducible mass introduced by Christodoulou in [32]. It is defined as

$$M_{\text{irr}} = \frac{1}{2}\left(M^2 + \sqrt{M^4 - \frac{J^2}{G^2}}\right) \tag{1.52}$$

Varying (1.52) we indeed obtain

$$\delta M_{\text{irr}} = \frac{1}{2\kappa G}\left(\delta M - \Omega \delta J\right) \tag{1.53}$$

where κ is the surface gravity for (1.41).

$$\kappa = \frac{r_+ - GM}{2GMr_+} \tag{1.54}$$

We see that (1.50) implies (1.51). This intuitively shows that according to the second law of black hole mechanics the area of the event horizon cannot decrease as it is straightforward to show that the irreducible mass is related to the area A of the black hole event horizon computed using (1.30).

$$M_{\text{irr}} = \frac{A}{16\pi} \tag{1.55}$$

Then we can conclude that the area can never decrease in this process.

$$\delta A \geq 0 \tag{1.56}$$

Moreover, as argued in [3], the irreducible mass can be interpreted as a sort of energy that cannot be converted into work. This just resembles the second law of thermodynamics, where increasing of entropy implies degradation of energy. The irreducible mass so represents a sort of degraded energy that cannot be extracted from the black hole by means of Penrose process.

1.3.1 The generalized second law

If we consider a situation in which an object with some common entropy enters a black hole, then it seems that the entropy of the universe has decreased, as an external observer cannot directly verify that it has not decreased. But we have seen that the area theorem states the area of the event horizon, which we identified with the black hole entropy, cannot decrease in any process. Therefore, Bekenstein [3] proposed a generalized version of the second law

The common entropy in the black hole exterior plus the black hole entropy never decreases

So black holes do contribute to the total amount of entropy in the universe.

1.4 The Third Law

The third law of black hole mechanics as stated in [8] says that we cannot reduce the surface gravity κ to zero in a finite amount of operations. This is equivalent to the third law of thermodynamics which states that we cannot reach zero temperature in a finite sequence of operations. We do not discuss it here, but details can be found in the references given above.

We conclude by summarizing the analogy between the four laws of black hole mechanics and those of thermodynamics.

Law	Thermodynamics	Black Holes
zeroth	T is constant at equilibrium	κ is constant at event horizon
first	$dE = TdS +$ work terms	$dM = \frac{\kappa}{8\pi} dA +$ work terms
second	$\delta S \geq 0$	$\delta A \geq 0$
third	$T = 0$ cannot be reached in a finite number of operations	$\kappa = 0$ cannot be reached in a finite number of operations

We shall stress again that at a classical level this just an analogy, the identification

$$S \leftrightarrow A, \qquad T \leftrightarrow \kappa$$

becomes exact when quantum effects are taken into consideration. It is the existence of Hawking radiation that turns a black hole into a real thermodynamic object.

2 Extremal Black Holes in $\mathcal{N} = 2$ Supergravity

2.1 $d = 4$, $\mathcal{N} = 2$ Gauged Supergravity

We are working in the framework of $d = 4$, $\mathcal{N} = 2$ supergravity, ([20], [34], [35]). At its classical level it represent a relativistic theory coupled with matter. $\mathcal{N} = 2$ refers to the number of supersymmetry generators and corresponds to 8 supercharges. The field content is organized in multiplets containing the same number of bosonic and fermionic degrees of freedom, as is required in any supersymmetric theory.

The multiplet content is given by

- The *supergravity multiplet* containing the graviton $g_{\mu\nu}$ (2 dofs on-shell), two gravitini Ψ^i_μ (two dofs on-shell each) and a massless vector, the graviphoton A^0_μ (2 dofs on-shell).

- One *vector multiplet* containing a massless vector A^i_μ (2 dofs on-shell), a pair of gauginos λ^i_μ (2 dofs on-shell each) and a complex scalar z (2 dofs on-shell). The scalar field parametrizes a special Kähler manifold. Details about quantities in special Kähler geometry are provided in appendix A. As a reference one can see [36].

The presence of the vector field allows for the possibility to gauge the theory. For the case at hand we are considering gauged supergravity with abelian Fayet-Ilioupoulos gauging.
The symmetry that is being gauged is a subgroup of the R-symmetry of the theory. An R-symmetry is a symmetry that commutes with Poincarè generators, but not with the supersymmetry charges. In $\mathcal{N} = 2$ the R-symmetry group is given by $SU(2) \times U(1)$. Fayet-Iliioupulos gauging involves gauging the $U(1)$ subgroup.

After the gauging procedure a scalar potential appears which accounts for AdS vacua solutions. The special Kähler geometry provides the coupling between the fields in the supergravity multiplet and in the vector multiplet. Since the gauge group is abelian, the scalar field is not charged, while only the two gravitinos acquire charge given by

$$e_\Lambda \equiv g\xi_\Lambda \tag{2.1}$$

where the ξ_Λ are constants called Fayet-Iliopoulos parameters.

We set the fermionic part of the lagrangian to zero, since we are interested in the bosonic part only.
Throughout the work we are going to set $\hbar = c = 8\pi G = 1$ and use the mostly minus signature convention, as in [2], which we follow closely in this chapter.

2.1.1 The Bosonic Lagrangian

The bosonic part of the supergravity lagrangian in 4 dimensions and $\mathcal{N} = 2$ with n_V vector multiplet and no hypermultiplets, is given by

$$\mathcal{L} = \frac{R}{2} + g_{i\bar{j}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} + \mathcal{I}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{2}\mathcal{R}_{\Lambda\Sigma}\epsilon^{\mu\nu\rho\sigma} F^\Lambda_{\mu\nu} F^\Sigma_{\rho\sigma} - V_g \tag{2.2}$$

with $\Lambda, \Sigma = 0, 1, ..., n_V$ and $i, j = 1, ..., n_V$. $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ are, respectively, the real and the imaginary part of the period matrix $\mathcal{N}$, defined in (A.7) in terms of symplectic sections, X^Λ and F_Λ, describing the Kähler manifold (as mentioned earlier, details are provided in A).
The fundamental bosonic fields are the metric $g_{\mu\nu}$, the $n_V + 1$ vector fields A^Λ_μ, where A^0_μ is called the graviphoton and n_V scalar fields. Here we are considering one vector multiplet,

$$n_V = 1 \tag{2.3}$$

Furthermore we set the electric charges to zero, allow only for magnetic charges and a radially varying scalar field, which is taken to be real. The symplectic sections are expressed in terms of a holomorphic function called the prepotential

$$F = -2i\sqrt{X^0(X^1)^3} \tag{2.4}$$

The lagrangian can then be written as

$$\mathcal{L} = \frac{R}{2} + g_{i\bar{j}}\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} + \mathcal{I}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} - V_g \tag{2.5}$$

Since the gauge group is abelian, the vector multiplet scalar field is neutral, and the only charged fields are the two gravitini.

The quantities $g_{i\bar{j}}, z, \mathcal{I}_{\Lambda\Sigma}, \mathcal{R}_{\Lambda\Sigma}, V_g$ are related with special geometry and can be written in terms of holomorphic symplectic sections (X^Λ, F_Λ). In fact the scalar fields parametrize a special Kähler manifold, which is described by holomorphic sections (see Appendix A).
We work in a static and spherically symmetric ansatz, [2]

$$ds^2 = U^2(r)dt^2 - \frac{1}{U^2(r)}dr^2 - h^2(r)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.6}$$

with

$$U^2(r) = e^{K(r)} f(r) \tag{2.7}$$

and

$$h^2(r) = e^{-K(r)} r^2 \tag{2.8}$$

The spherical symmetry, Bianchi identity and Maxwell equation, constrain the form of the field strength to be

$$F^\Lambda_{tr} = -\frac{1}{2h^2(r)}\mathcal{I}^{\Lambda\Sigma}\left(\mathcal{R}_{\Sigma\Gamma}p^\Gamma - q_\Sigma\right) \tag{2.9}$$

$$F^\Lambda_{\theta\phi} = \frac{1}{2}p^\Lambda \sin\theta \tag{2.10}$$

The electric q_Λ and magnetic p^Λ charges are given by

$$p^\Lambda = -\frac{1}{4}\int_{S^2} F^\Lambda \tag{2.11}$$

$$q_\Lambda = -\frac{1}{4}\int_{S^2} G_\Lambda \tag{2.12}$$

where G_Λ is the dual field strength defined as

$$G_{\Lambda\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial F^\Lambda_{\rho\sigma}} = \mathcal{R}_{\Lambda\Sigma}F^\Sigma_{\mu\nu} - \frac{1}{2}\mathcal{I}_{\Lambda\Sigma}\epsilon_{\mu\nu\rho\sigma}F^{\Sigma\rho\sigma} \tag{2.13}$$

2.2 Magnetic Configuration

The first example of BPS black holes with spherical horizon in the context of gauged supergravity was treated by Cacciatori and Klemm, [20], finding solutions to the set of equation in [37], and then analyzed in [34] and [35]. In [2], the supersymmetric solutions of [20] are deformed to non-extremal configurations, setting the warp factor $f(r)$ to be

$$f(r) = \kappa + \frac{c_1}{r} + \frac{c_2}{r^2} + \tilde{g}^2 r^2 e^{-2K(r)} \tag{2.14}$$

where $\kappa = 1$ for spherical configurations and $\tilde{g} = g\tilde{\xi}$.

$$\tilde{\xi} = \frac{\hat{\xi}}{\beta}, \qquad \hat{\xi} = \frac{\sqrt{2}\xi_0^{1/4}\xi_1^{3/4}}{3^{3/4}} \tag{2.15}$$

where β is a non physical parameter explicitly expressed here to make it easier to compare different results in literature. Later on it will be set to one.
The field strength is only magnetic

$$F^\Lambda_{tr} = 0, \qquad F^\Lambda_{\theta\phi} = \frac{p^\Lambda}{2}\sin\theta \tag{2.16}$$

The scalar field is expressed as

$$z = \frac{X^1}{X^0} = \frac{H^1}{H^0} \tag{2.17}$$

with [1]

$$H^\Lambda = \frac{X^\Lambda + \bar{X}^\Lambda}{2} \tag{2.18}$$

and the sections H^Λ are expressed in terms of harmonic functions

$$H_\Lambda = a_\Lambda + \frac{b_\Lambda}{r} \tag{2.19}$$

The Kähler potential is

$$\begin{aligned} e^{-K(r)} &= \beta^2\sqrt{H_0(H_1)^3} \\ &= \beta^2\sqrt{\left(a_0 + \frac{b_0}{r}\right)\left(a_1 + \frac{b_1}{r}\right)^3} \\ &= \beta^2\frac{\sqrt{(r+Q_0)\,(r+Q_1)^3}}{r^2} \end{aligned} \tag{2.20}$$

where we have defined

$$Q_0 = \frac{a_0}{b_0}, \qquad Q_1 = \frac{a_1}{b_1} \tag{2.21}$$

and used the property[2] $a_0 a_1^3 = 1$ and set β to one, since it is not a physical parame-

[1]The X^Λ sections are real, therefore $X^\Lambda = \bar{X}^\Lambda$

[2]Einstein's equations put constraints on the a_Λ parameters imposing that

$$a_0 = \frac{\sqrt{2}g\xi_1^{3/2}l_{AdS}}{3\sqrt{3}\sqrt{\xi_0}} = \left(\frac{\xi_1^3}{27\xi_0^3}\right)^{1/4} \tag{2.22}$$

$$a_1 = \frac{\sqrt{2}g\sqrt{\xi_0\xi_1}l_{AdS}}{\sqrt{3}} = \left(\frac{3\xi_0}{\xi_1}\right)^{1/4} \tag{2.23}$$

We observe that

$$a_0 a_1^3 = 1 \tag{2.24}$$

ter. Thanks to the scaling symmetry

$$r \to \lambda r, \quad c_1 \to \lambda c_1, \quad c_2 \to \lambda^2 c_2, \quad b_\Lambda \to \lambda b_\Lambda, \quad \beta \to \frac{\beta}{\lambda} \tag{2.25}$$

we can safely set $\beta = 1$.

2.2.1 AdS asymptotics

We are looking for solution that are asymptotically AdS, so that the scalar potential has to reproduce a negative cosmological constant Λ at infinity. The scalar potential is given by

$$V_g = -g^2 \left(\frac{\xi_0 \xi_1}{\sqrt{z}} + \frac{\xi_1^2}{3}\sqrt{z} \right) \tag{2.26}$$

$$\frac{\partial}{\partial z}\left[-g^2 \left(\frac{\xi_0 \xi_1}{\sqrt{z}} + \frac{\xi_1^2}{3}\sqrt{z} \right)\right] = -g^2 \left(-\frac{\xi_0 \xi_1}{2\sqrt{z^3}} + \frac{\xi_1^2}{6\sqrt{z}} \right) \tag{2.27}$$

$$\partial_z V_g|_\infty = 0 \implies z_\infty = \frac{3\xi_0}{\xi_1} \tag{2.28}$$

Since in an AdS_{d+1} spacetime the cosmological constant is related to the dimension of the spacetime via

$$\Lambda = -\frac{d(d-1)}{2l^2_{AdS}} \tag{2.29}$$

we have that

$$V_g(z_\infty) = -\frac{3}{l^2_{AdS}} = \Lambda \tag{2.30}$$

with

$$l^2_{Ads} = \frac{1}{g^2 \sqrt{\frac{4}{27}\xi_0 \xi_1^3}} \tag{2.31}$$

Superpotential

Whenever the gauging potential can be written as (for our real scalar field this is the case)

$$V_g = g^2 \left(-3\mathcal{W}^2 + g^{ij}\partial_i \mathcal{W} \partial_j \mathcal{W} \right) \tag{2.32}$$

where $\mathcal{W}$ is called the superpotential function, the radial evolution of the scalar field can be written in terms of $\mathcal{W}$, ([2], [35]), as

$$z^{i'} = -\frac{e^{K/2}}{\tilde{\xi} r} g^{ij}\partial_j \mathcal{W}, \qquad \left(r e^{-K/2} \right)' = \frac{\mathcal{W}}{\tilde{\xi}} \tag{2.33}$$

where $\tilde{\xi}$ is as defined before.

The superpotential will play an important role in the counterterm analysis for the holographic renormalization procedure of the subsequent chapters. We do not discuss here the details about the introduction and the origin of the superpotential, that can be found in both [2] and [35].

2.2.2 Magnetic charges

The magnetic charges p^Λ are derived from the Hamiltonian constraint, which has to be imposed when rewriting the action as a sum of squares, as explained in [2]

$$V_{BH} = -\frac{1}{2} p^\Lambda \mathcal{I}_{\Lambda\Sigma} p^\Sigma = -2\left(\tilde{b}^\Lambda \mathcal{I}_{\Lambda\Sigma} \tilde{b}^\Sigma + c_2 \tilde{a}^\Lambda \mathcal{I}_{\Lambda\Sigma} \tilde{a}^\Sigma - c_1 \tilde{b}^\Lambda \mathcal{I}_{\Lambda\Sigma} \tilde{a}^\Sigma\right) \tag{2.34}$$

where $\mathcal{I}_{\Lambda\Sigma}$ is as defined in A.2 and the tilded parameters are related to the untilded ones by

$$\tilde{a}_\Lambda = \frac{a_\Lambda}{2\sqrt{2}}, \qquad \tilde{b}_\Lambda = \frac{b_\Lambda}{2\sqrt{2}} \tag{2.35}$$

Solving (2.34) leads us to the following values for the magnetic charges

$$p^0 = \pm \frac{\sqrt{b_0(b_0 - c_1 a_0) + c_2 a_0^2}}{\sqrt{2}} \tag{2.36}$$

$$p^1 = \pm \frac{\sqrt{b_1(b_1 - c_1 a_1) + c_2 a_1^2}}{\sqrt{2}} \tag{2.37}$$

where c_1 and c_2 are real parameters comparing in the warp factor $f(r)$.
These are the magnetic charges for the non extremal configuration set by (2.14). In what follows we show how to obtain the extremal configuration which is also supersymmetric.

2.2.3 Extremal BPS Solutions

In the case of magnetically charged black holes with running scalars we can have extremal and supersymmetric black holes. The existence of a horizon shielding the black hole singularity is related to non trivial scalar profiles. Constant scalars lead to naked singularities, [38].
The extremality condition is verified when the warp factor $f(r)$ presents a double pole. In other words when $f(r)$ can be written as

$$f(r) = \frac{1}{r^2 l_{AdS}^2}\left(r^2 - a^2\right)^2 \tag{2.38}$$

where $a = r_h$ is the horizon radius.
Supersymmetric solutions have also to satisfy a constraint on the charges, known as the Dirac quantization condition

$$g_\Lambda p^\Lambda = \pm 1 \tag{2.39}$$

Imposing that the warp factor $f(r)$ should be written as (2.38), puts a series of conditions on the parameters c_1 and c_2, with which the warp factor has been defined. We can write $f(r)$ as a perfect square plus other terms as follows

$$\begin{aligned} f(r) = \frac{1}{r^2 l_{AdS}^2}\left(r^2 - \left(3Q_1^3 - \frac{l_{AdS}^2}{2}\right)\right)^2 + \\ + \frac{1}{r}\left(c_1 - \frac{8Q_1^3}{l_{AdS}^2} + \frac{1}{r^2}\left(c_2 - \frac{12Q_1^4}{l_{AdS}^2} - \frac{l_{AdS}^2}{4} + 3Q_1^2\right)\right) \end{aligned} \tag{2.40}$$

Imposing (2.38) produces the following BPS limits for c_1 and c_2, when the Dirac-like quantization supersymmetric condition is chosen to be

$$g_\Lambda p^\Lambda = -1 \tag{2.41}$$

$$c_1 = \frac{8}{l_{AdS}^2} Q_1^3 \beta^2 \tag{2.42}$$

$$c_2 = -3Q_1^2 + \frac{l_{AdS}^2}{4\beta^2} + \frac{12}{l_{AdS}^2}\beta^2 Q_1^4 \tag{2.43}$$

The horizon is then given by

$$r_h = a = \sqrt{3Q_1^2 - \frac{l_{AdS}^2}{2}} \tag{2.44}$$

In the extremal limit the magnetic charges p^Λ can be evaluated from (2.36) and (2.37) using the extremal limits for the parameters c_1 and c_2, (2.42) and (2.43). We obtain

$$p^0 = \pm\left(\frac{1}{4g\xi_0} + \frac{2}{3} g b_1^2 \frac{\xi_1^2}{\xi_0}\beta\right) \tag{2.45}$$

$$p^1 = \pm\left(-\frac{3}{4}\frac{1}{g\xi_1} + \frac{2}{3} g \xi_1 b_1^2 \beta^2\right) \tag{2.46}$$

Upon imposing the Dirac quantization (2.41), we have to choose the minus sign for p^0 and the plus sign for p^1.

$$p^0_{extr} = -\frac{1}{4g\xi_0} - \frac{2}{3}gb_1^2\frac{\xi_1^2}{\xi_0}; \qquad p^1_{extr} = -\frac{3}{4}\frac{1}{g\xi_1} + \frac{2}{3}g\xi_1 b_1^2 \tag{2.47}$$

The other two solutions are extremal but not supersymmetric.
In an analogous way we can find the extremal limit for the magnetosatic potential, defined as the value of the dual electric field at infinity upon the assumption that it vanishes on the horizon r_+

$$\chi_\Lambda = -\int_{r_+}^{\infty} G_{\Lambda,tr} dr \tag{2.48}$$

where G_Λ is defined as in (2.13), which in our case becomes

$$G_{\Lambda,tr} = \frac{e^K}{2r^2}\mathcal{I}_{\Lambda\Sigma}p^\Sigma \tag{2.49}$$

We have that

$$G_{0,tr} = \frac{1}{\sqrt{\left(a_0 + \frac{b_0}{r}\right)\left(a_1 + \frac{b_1}{r}\right)^3}}\frac{1}{2r^2}\left(-z^{3/2}\right)p^0 = -\frac{1}{2}\frac{p^0}{(a_0 r + b_0)^2} \tag{2.50}$$

and

$$G_{1,tr} = \frac{1}{\sqrt{\left(a_0 + \frac{b_0}{r}\right)\left(a_1 + \frac{b_1}{r}\right)^3}}\frac{1}{2r^2}\left(-\frac{3}{\sqrt{z}}\right)p^1 = -\frac{3}{2}\frac{p^1}{(a_1 r + b_1)^2} \tag{2.51}$$

where we have used (2.17).
Integrating from r_+ to infinity, we get

$$\chi_0 = \frac{1}{2}\frac{p^0}{a_0\left(a_0 r_+ + b_0\right)}, \qquad \chi_1 = \frac{3}{2}\frac{p^1}{a_1\left(a_1 r_+ + b_0\right)} \tag{2.52}$$

Upon rewriting we end up with

$$\chi_0 = \frac{p^0}{2a_0^2\left(\frac{b_0}{a_0} + r_+\right)} = g_0^2 l_{AdS}^2 \frac{p^0}{r_+ - 3Q_1} \tag{2.53}$$

$$\chi_1 = \frac{3p^1}{2a_1^2\left(\frac{b_1}{a_1} + r_+\right)} = g_1^2 l_{AdS}^2 \frac{p^1}{3(Q_1 + r_+)} \tag{2.54}$$

Taking the extremal limit of (2.53) and (2.54) means evaluating them at the extremal values for the magnetic charges as found in (2.47).

We have re-derived here the BPS solution of [2], which will prove useful in the following chapters when discussing the thermodynamic properties. At last, we give the expressions for temperature and entropy.

Temperature and Entropy

We compute the temperature T and the entropy S as from equation derived, respectively, in (1.15) and (1.39) for our magnetic black hole.
The temperature is then

$$T = \frac{1}{4\pi} e^{K(r)} \frac{df(r)}{dr}\bigg|_{r=r_+} \tag{2.55}$$

Using the radial derivative of the $f(r)$ warp function as in (B.21), evaluated at the outer horizon r_+

$$T = \frac{1}{4\pi} e^{K(r_+)} \frac{c_1 l_{AdS}^2 - 8Q_1^3 + 2l_{AdS}^2 r_+ - 12Q_1^2 r_+ + 4r_+^3}{l_{AdS}^2 r_+^2} \tag{2.56}$$

where we have used the defining equation for r_+ to explicit c_2

$$c_2 = -(c_1 + r_+)r_+ - \frac{1}{l_{AdS}^2}\left(r_+ - 3Q_1\right)\left(r_+ + Q_1\right)^3 \tag{2.57}$$

On the other side the entropy is the area of the event horizon as defined in (1.30)

$$\begin{aligned} S &= \frac{A}{4} = \\ &= \frac{1}{4}\left(\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{g_{\theta\theta} g_{\phi\phi}}\right)\bigg|_{r_+} = \\ &= \frac{1}{4}\left(\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{e^{-2K(r)} r^4 \sin^2(\theta)}\right)\bigg|_{r_+} = \\ &= \frac{1}{4}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi e^{-K(r_+)} r_+^2 \sin(\theta) = \\ &= \frac{1}{2}\pi r_+^2 e^{-K(r_+)} \left[-\cos(\theta)\right]_0^{\pi} = \\ &= \pi r_+^2 e^{-K(r_+)} \end{aligned} \tag{2.58}$$

Therefore

$$S = \pi r_+^2 e^{-K(r_+)} \tag{2.59}$$

2.3 Scalar Field Analysis

Here we give a brief analysis of the scalar field. We describe the reparametrization to rewrite it as a canonically renormalized field, study its expansion at the asymptotic infinity and discuss the allowed boundary conditions.
For details on scalar field expansion in AdS spacetime and boundary conditions one can see [23], [24], [39].

2.3.1 Defining a canonical scalar field

For the calculations in the next section, it is useful to reparametrize the scalar field z as follows.

$$z = e^{x\phi(r)+y} \tag{2.60}$$

where x and y are constants and ϕ is the canonically renormalized field. The constant y allows to choose a reference value of the field, which is chosen to the asymptotic infinity ($r \to \infty$)

$$\varphi(r) = \phi(r) - \phi(\infty) \tag{2.61}$$

If we want ϕ and thus φ be a canonical normalized field, then the kinetic term should be of the form

$$g_{i\bar{j}}\partial_\mu z^i \partial^\mu z^{\bar{j}} \to \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi \tag{2.62}$$

Using (A.31) for the metric $g_{i\bar{j}}$

$$g_{i\bar{j}}\partial_\mu z^i \partial^\mu z^{\bar{j}} = \frac{3}{16z^2}\partial_\mu z \partial^\mu z \tag{2.63}$$

and

$$z = e^{x(\varphi+\phi(\infty))+y} \tag{2.64}$$

and imposing (2.62)

$$\frac{3}{16z^2}x^2\partial_\mu \varphi \partial^\mu \varphi z^2 = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi \tag{2.65}$$

we obtain

$$x^2 = \frac{8}{3} \tag{2.66}$$

Then

$$z(r) = z_\infty e^{\sqrt{8/3}\varphi(r)} \tag{2.67}$$

With this ansatz for the scalar field the lagrangian (2.5) becomes

$$\begin{aligned}\mathcal{L} =& \frac{R}{2} + \frac{1}{2}\partial_\mu \varphi(r)\partial^\mu \varphi(r) - \left(\frac{3\xi_0}{\xi_1}\right)^{3/2} e^{\sqrt{6}\varphi} F^0_{\mu\nu}F^{0\mu\nu} + \\ &- 3\left(\frac{3\xi_0}{\xi_1}\right)^{-1/2} e^{-\sqrt{2/3}\varphi} F^1_{\mu\nu}F^{1\mu\nu} - V_g(\varphi)\end{aligned} \tag{2.68}$$

and the scalar potential (2.26) becomes

$$\begin{aligned}V_g &= -g^2\left(\frac{\xi_0\xi_1}{\sqrt{z}} + \frac{\xi_1^2}{3}\sqrt{z}\right) = \\ &= -g^2\left(\xi_0\xi_1\left(\frac{\xi_1}{3\xi_0}\right)^{1/2} e^{-\sqrt{2/3}\varphi} + \frac{\xi_1^2}{3}\left(\frac{3\xi_0}{\xi_1}\right)^{1/2} e^{\sqrt{2/3}\varphi}\right) = \\ &= -2g^2\left(\frac{1}{3}\xi_0\xi_1^3\right)^{1/2} \frac{\left(e^{-\sqrt{2/3}\varphi} + e^{\sqrt{2/3}\varphi}\right)}{2}\end{aligned} \tag{2.69}$$

which can be written as

$$V_g(\varphi) = -\frac{3}{l_{AdS}^2}\cosh\left(\sqrt{\frac{2}{3}}\varphi\right) \tag{2.70}$$

This is what has been obtained in [2]. The cosh potential was analyzed in detail in [24].

Since we are working in an asymptotically AdS spacetime, for which

$$V_g^{'}(\varphi(\infty)) = 0; \qquad V_g(\varphi(\infty)) = -\frac{d(d-1)}{2l_{AdS}^2} \tag{2.71}$$

we can expand[3] the scalar potential in the vicinity of the extremum as follows

$$V_g(\varphi) = -\frac{d(d-1)}{2l_{AdS}^2} + \frac{1}{2}m^2(\varphi - \varphi_\infty)^2 + o(\varphi - \varphi_\infty)^2 \tag{2.72}$$

We can read the scalar field mass

$$\begin{aligned}V_g(\varphi) &= -\frac{3}{l_{AdS}^2}\cosh\left(\sqrt{\frac{2}{3}}\varphi\right) = \\ &= -\frac{3}{l_{AdS}^2}\left(1 + \frac{1}{2}\frac{2}{3}\varphi^2\right) = \\ &= -\frac{3}{l_{AdS}^2} + \frac{1}{2}\left(-\frac{2}{l_{AdS}^2}\right)\varphi^2\end{aligned} \tag{2.73}$$

[3]Reminder of the expansion

$$\cosh(x) = 1 + \frac{x^2}{2} + ...$$

Therefore

$$m^2 = -\frac{2}{l_{AdS}^2} \tag{2.74}$$

The mass satisfies the Breitenlohner-Friedman bound (see [23], [24])

$$m^2 l_{AdS}^2 \geq -\left(\frac{d}{2}\right)^2 \tag{2.75}$$

where $d = 3$ in our analysis.
In particular it satisfies a more constraining condition, ([23])

$$-\left(\frac{d}{2}\right)^2 \leq m^2 l_{AdS}^2 \leq -\left(\frac{d}{2}\right)^2 + 1 \tag{2.76}$$

which allows for Neumann and mixed boundary conditions at the asymptotic infinity.

2.3.2 Boundary Conditions on Scalar Field

In order to analyze the boundary conditions to be imposed on the canonically renormalized field φ, we should analyze its expansion at the conformal boundary.

Metric and field expansion on the boundary

Asymptotically the metric ansatz we are using can be written as

$$ds_\infty^2 = -\frac{l_{AdS}^2}{r^2} dr^2 + \frac{r^2}{l_{AdS}^2} h_{0,ij} dx^i dx^j \tag{2.77}$$

where h_0 is the minkowski metric on the boundary. We indicate the anti-de Sitter spacetime $\mathcal{M}$ and its conformal boundary by $\partial\mathcal{M}$, which is located at $r = \infty$. We are going to discuss this in the next chapter.

As in [2], the metric on the boundary h_0 is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -l_{AdS}^2 & 0 \\ 0 & 0 & -l_{AdS}^2 \sin^2\theta \end{pmatrix} \tag{2.78}$$

After defining

$$\frac{r}{l_{AdS}} = e^{\frac{\tilde{r}}{l_{AdS}}} \implies d\tilde{r}^2 = \frac{l_{AdS}^2}{r^2} \tag{2.79}$$

we can rewrite (2.77) to match the conventions in [2] and [24].

$$ds_\infty^2 = -d\tilde{r}^2 + e^{-2\tilde{r}/l_{AdS}} h_{0,ij} dx^i dx^j \tag{2.80}$$

We then rename $\tilde{r} = r$.

On the other hand, having written the scalar filed z in terms of the canonically normalized filed φ as in (A.51) and having imposed the vanishing at $r \to \infty$ for the φ field, we can expand this last as

$$\varphi(r) \sim \frac{\alpha_1}{r^{\Delta_-}} + \frac{\alpha_2}{r^{\Delta_+}} + ... \tag{2.81}$$

with [4]

$$\alpha_1 = \frac{\sqrt{6} b_1}{a_1}; \qquad \alpha_2 = \frac{\alpha_1^2}{\sqrt{6}} \tag{2.82}$$

with Δ_- and Δ_+ the solutions of

$$m^2 l_{AdS}^2 = \Delta(\Delta - d) \tag{2.83}$$

We have

$$\Delta_- = 1; \qquad \Delta_+ = 2 \tag{2.84}$$

On the other hand,if we denote by φ_- and φ_+ the two modes, solutions of the Klein-Gordon equation for the scalar field, then the scalar field can be expanded as

$$\varphi(r) \sim e^{-\Delta_- r/l_{AdS}} \varphi_- + e^{-\Delta_+ r/l_{AdS}} \varphi_+ \tag{2.85}$$

[4]To find the coefficients of the expansion in (2.81), we proceed expanding the z field defined in terms of symplectic sections

$$z(r) = \frac{X_1}{X_0} = \frac{a_1 r + b_1}{a_0 r + b_0}$$

up to second order in $1/r$ as follows:

$$z(r) = \frac{a_1 r + b_1}{a_0 r + b_0} = \frac{a_0 r \left(\frac{a_1}{a_0} + \frac{b_1}{a_0 r}\right)}{a_0 r \left(1 + \frac{b_0}{a_0 r}\right)} = \left(\frac{a_1}{a_0} + \frac{b_1}{a_0 r}\right)\left(1 - \frac{b_0}{a_0 r} + \frac{b_0^2}{a_0^2 r^2}\right) =$$
$$= \frac{a_1}{a_0}\left(1 + \frac{4Q_1}{r} + \frac{12 Q_1^2}{r^2} + O(r^{-3})\right)$$

where we have used the definitions (A.47) and the relation $Q_0 = -3Q_1$. Using the definitions (2.22) and (2.23) of a_0 and a_1, we find that $a_1/a_0 = z_\infty$.
On the other side, expanding up to the same order (A.51) using the φ expansion of (2.81)

$$z(r) = z_\infty \left(1 + \sqrt{\frac{8}{3}}\frac{\alpha_1}{r} + \left(\sqrt{\frac{8}{3}}\alpha_2 + \frac{4}{3}\alpha_1^2\right)\frac{1}{r^2} + O(r^{-3})\right)$$

and comparing the coefficients of the expansions, we get the relations (2.82).

Comparing (2.81) and (2.85), we can explicitly write the two modes

$$\varphi_- = \frac{\sqrt{6}Q_1}{l_{AdS}}; \qquad \varphi_+ = \frac{\sqrt{6}Q_1^2}{l_{AdS}^2} \tag{2.86}$$

In [24] the boundary terms required to impose different boundary conditions are discussed throughout. We report here here the the boundary terms that should be added to the lagrangian in order to satisfy Neumann and mixed boundary conditions.
For Neumann boundary conditions we have

$$S_{\text{Neumann}} = -\int_{\partial\mathcal{M}} d^3x \sqrt{h_0}\varphi_- \hat{\pi}_{(\Delta_+)} \tag{2.87}$$

whereas for mixed boundary condition the additional term takes the form

$$S_{\text{mixed}} = S_{\text{Neumann}} + \int_{\partial\mathcal{M}} d^3x \sqrt{h_0} \left(f(\varphi_-) - \varphi_- f'(\varphi_-) \right) \tag{2.88}$$

where h_0 is as defined in (2.78) and $\hat{\pi}_{(\Delta_+)}$ is defined in [24] as

$$\hat{\pi}_{\Delta_+} = \lim_{r\to\infty} e^{\frac{\Delta_+}{r} l_{AdS}} \pi_{(\Delta_+)} \tag{2.89}$$

with $\pi_{(\Delta_+)}$ the momentum conjugated to φ_-. In [24] it is argued why we have to use the conjugate momentum and not φ_- to impose boundary conditions. Briefly the reason lies in the fact that $\pi_{(\Delta_+)}$ behaves well under radial shifts. This is crucial since we are going to work in a truncated spacetime at some r_0 and then we are going to push the boundary to $r = \infty$, (see chapter 3).
For our solution (see [2])

$$\pi_{(\Delta_+)} = \frac{2}{l_{AdS}} e^{-2r/l_{AdS}} \varphi_+, \qquad f'(\varphi_-) = -\hat{\pi}_{(\Delta_+)} \tag{2.90}$$

3 Mass in AdS and Holographic Renormalization

Thanks to the AdS/CFT correspondence we can relate a string theory defined in the bulk of AdS spacetime to a conformal quantum field theory which lives on the conformal boundary of AdS spacetime. String theory can be approximated by supergravity, a weakly coupled theory. Therefore the correspondence is between weakly coupled gravitational theory and strongly coupled quantum field theory.
In particular the correspondence allows to identify the radial coordinate of the gravitational theory with the renormalization group scale of the boundary theory. This is the idea at the basis of holographic renormalization. In fact, the UV divergencies, which come up in filed theories, correspond to IR (large r) divergences of the gravitational theory. Therefore, quantities in the filed theory are renormalized by removing the large r divergencies.

In the holographic renormalizion procedure we consider a radial foliation of the spacetime, in other words we divide it into hypersurfaces of constant r and take them isomorphic to the boundary. In the subsequent computations, to manage the infrared divergencies near the boundary, we will consider a truncated AdS spacetime by putting a cut-off at a finite radial coordinate r_0 and then taking the limit to $r \to \infty$, as the setup in [2]. Therefore all the calculations will be performed in the truncated spacetime. Since AdS spacetime possesses a boundary, which is a 3-dimensional hypersurface embedded in 4-dimensional spacetime, we will need the following notions of induced metric and extrinsic curvature.

In the first part of this chapter we are going to define the concept of extrinsic curvature and explicitly perform the variation of the gravitational action. In second part, we compute the mass of the black hole from the tt component of the stress-energy tensor and compare the different prescriptions for both finite terms and counterterms, as given in the two works, [1] and [2], main object of our project.

3.1 Projection operator on a hypersurface

Considering the boundary as a hypersurface $\partial\mathcal{M}$ embedded in the spacetime manifold which we denote with $\mathcal{M}$, we take n^μ to be the unit normal spacelike vector to $\partial\mathcal{M}$ satisfying

$$n^\mu n_\mu = -1 \tag{3.1}$$

We can defined a projector operator to project quantities living in the ambient space $\mathcal{M}$ onto the hypersurface $\partial\mathcal{M}$

$$h^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu \tag{3.2}$$

It can be straightforwardly checked that h^μ_ν is a projector operator, since it satisfies the following two properties

- It is orthogonal to the hypersurface normal n^μ

$$\begin{aligned} n^\mu h^\nu_\mu &= \left(\delta^\nu_\mu + n^\nu n_\nu\right) n^\mu = \\ &= \delta^\nu_\mu n^\mu + \underbrace{\left(n^\mu n_\mu\right)}_{-1} n^\nu = \\ &= n^\nu - n^\nu = \\ &= 0 \end{aligned} \tag{3.3}$$

- It is idempotent, i.e. it squares to the identity operator

$$\begin{aligned} h^\mu_\lambda h^\lambda_\nu &= \left(\delta^\mu_\lambda + n^\mu n_\lambda\right)\left(\delta^\lambda_\nu + n^\lambda n_\nu\right) = \\ &= \delta^\mu_\nu + n^\mu n_\nu + n^\mu n_\nu + n^\mu n_\lambda n^\lambda n_\nu = \\ &= \delta^\mu_\nu + 2n^\mu n_\nu - n^\mu n_\nu = \\ &= \delta^\mu_\nu + n^\mu n_\nu = \\ &= h^\mu_\nu \end{aligned} \tag{3.4}$$

3.1.1 Induced metric on the hypersurface

Starting with (3.2) we can use the metric tensor $g_{\mu\nu}$ to lower the upper index and obtain

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{3.5}$$

Note that given two vectors v^μ and u^μ tangent the hypersurface $\partial\mathcal{M}$, that is normal to n^μ, we get

$$\begin{aligned} g_{\mu\nu}v^\mu u^\nu &= \left(h_{\mu\nu} - n_\mu n_\nu\right) v^\mu u^\nu = \\ &= h_{\mu\nu}v^\mu u^\nu + \underbrace{n_\mu v^\mu}_{0}\underbrace{n_\nu u^\nu}_{0} = \\ &= h_{\mu\nu}v^\mu u^\nu \end{aligned} \tag{3.6}$$

This justifies considering $h_{\mu\nu}$ as the induced metric on the hypersurface. It is also known as the *first fundamental form*.
$h_{\mu\nu}$ has its components on the hypersurface and therefore projects any vector in $\mathcal{M}$ into a vector tangent to the hypersurface. Given a vector v^μ, the action of $h_{\mu\nu}$ on it will be orthogonal to n^μ.

$$\begin{aligned} \left(h_{\mu\nu}v^\mu\right) n^\nu &= g_{\mu\nu}v^\mu n^\nu + n_\mu v^\mu \underbrace{n_\nu n^\nu}_{-1} = \\ &= v^\mu n_\mu - v^\mu n_\mu = \\ &= 0 \end{aligned} \tag{3.7}$$

The inverse of $h_{\mu\nu}$ is given by $h^{\mu\nu}$

$$h^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu \tag{3.8}$$

and using (3.4) it satisfies the following property

$$\begin{aligned} h^{\alpha\beta}h_{\beta\gamma} &= h^\beta_\sigma g^{\sigma\alpha} h^\rho_\beta g_{\rho\gamma} = \\ &= h^\rho_\sigma g^{\sigma\alpha} g_{\rho\gamma} = \\ &= h^\alpha_\gamma \end{aligned} \tag{3.9}$$

3.2 Extrinsic curvature

For a hypersurface embedded in $\mathcal{M}$ it does make sense to talk about extrinsic curvature which depends on how the hypersurface is embedded in the ambient space. We remind that intrinsic curvature is the one measured by the Riemann tensor. We are following the definition used in [2]

$$\Theta_{\mu\nu} = -\frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right) \tag{3.10}$$

which comes from the geometric interpretation as explained in the following paragraph.

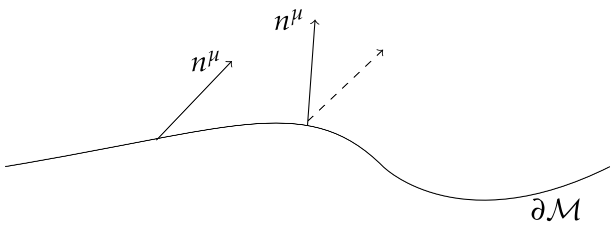
As shown, respectively in (D.10) and (D.11), $\Theta_{\mu\nu}$ is orthogonal to the normal vector n^μ and a symmetric rank 2 tensor.

3.2.1 Geometric interpretation

The extrinsic curvature measure how the hypersurface is embedded in the ambient space. For reference one can see [25], [26], [33]. Suppose we have a vector field v^μ in $\mathcal{M}$ which is parallely transported along another vector field u^μ so that

$$\nabla_v u = v^\mu \nabla_\mu u^\nu = 0 \tag{3.11}$$

holds. This relation is, in general, not true when projected on the hypersurface $\partial\mathcal{M}$ due to the embedding.



The extrinsic curvature is a $(0,2)$ tensor that measures how parallel transport fails on the hypersurface because of its bending in $\mathcal{M}$. We are using the definition of [33]

$$\begin{aligned} \Theta\colon \mathcal{T}_p(\mathcal{M}) \times \mathcal{T}_p(\mathcal{M}) &\to \mathbb{R} \\ v, u &\to n_\mu \left(h^\gamma_\beta v^\beta \nabla_\gamma \left(h^\mu_\rho u^\rho \right) \right) \end{aligned} \tag{3.12}$$

where n^μ is the unit normal vector to the hypersurface as before.
To find $\Theta_{\mu\nu}$ we should perform a little algebra on the above expression getting to

$$\begin{aligned} n_\mu \left(h^\gamma_\beta v^\beta \nabla_\gamma \left(h^\mu_\rho u^\rho \right) \right) &= - \left(h^\gamma_\beta v^\beta \right) \left(h^\mu_\rho u^\rho \right) \nabla_\gamma n_\mu = \\ &= -h^\gamma_\beta h^\mu_\rho \nabla_\gamma n_\mu \left(v^\beta u^\rho \right) \end{aligned} \tag{3.13}$$

where we have used the fact that $h^\mu_\rho u^\rho$ is tangent to the hypersurface so that $n_\mu h^\mu_\rho u^\rho = 0$ and therefore the first equality holds.

Then since

$$\Theta(v, u) = \Theta_{\mu\nu} v^\mu u^\nu \tag{3.14}$$

we have that

$$\Theta_{\mu\nu} = -h^\alpha_\mu h^\beta_\nu \nabla_\alpha n_\beta \tag{3.15}$$

In appendix D we show that there are different equivalent definitions of the extrinsic curvature tensor. Here we list them all:

$$
\begin{aligned}
\Theta_{\mu\nu} &= -h_\mu^\alpha h_\nu^\beta \nabla_\alpha n_\beta = \\
&= -h_\mu^\alpha \nabla_\alpha n_\nu = \\
&= -\nabla_\mu n_\nu = \\
&= -\frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right) = \\
&= -\frac{1}{2} h_\mu^\alpha h_\nu^\beta \mathcal{L}_n g_{\alpha\beta} = \\
&= -\frac{1}{2}\mathcal{L}_n h_{\mu\nu}
\end{aligned}
\tag{3.16}
$$

3.3 Variation of the gravitational action with boundary term

Since AdS spacetime possesses a conformal boundary we have to add the Gibbons-Hawking boundary term

$$
\begin{aligned}
S_{\text{grav}} &= S_{\text{bulk}} + S_{GH} = \\
&= \int d^4x\sqrt{-g}\left(\frac{R}{2} + g_{zz}\partial_\mu z \partial^\mu z + \mathcal{I}_{\Lambda\Sigma} F_{\mu\nu}^\Lambda F^{\Sigma\mu\nu} - V_g\right) - \int d^3x\sqrt{h}\Theta
\end{aligned}
\tag{3.17}
$$

Varying S_{grav} we obtain a bulk part given by the equation of motion as derived in (C.7) and a boundary term given by

$$
\delta S_{\text{bdy}} = \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h} n_\mu J^\mu - \int_{\partial_0\mathcal{M}} d^3x \delta\sqrt{h}\Theta - \int_{\partial_0\mathcal{M}} d^3x\sqrt{h}\delta\Theta \tag{3.18}
$$

The last two terms come from varying the Gibbons-Hawking term, whereas the first one comes from varying the Ricci tensor, which produces a divergence term. Let's analyze each term separately.

3.3.1 Divergence term from Ricci tensor

The J^μ term comes from the variation of the Ricci tensor

$$
R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu} \tag{3.19}
$$

$$
\delta R_{\mu\nu} = \partial_\rho \delta\Gamma^\rho_{\mu\nu} - \partial_\nu \delta\Gamma^\rho_{\rho\mu} + \delta\Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} + \Gamma^\rho_{\rho\lambda}\delta\Gamma^\lambda_{\mu\nu} - \delta\Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\rho\mu} \tag{3.20}
$$

Now we add and subtract the same terms in order to reconstruct the following covariant derivatives

$$
\nabla_\rho \delta\Gamma^\rho_{\mu\nu} = \partial_\rho \delta\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\rho\lambda}\delta\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\rho\mu}\delta\Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\rho\nu}\delta\Gamma^\rho_{\mu\lambda} \tag{3.21}
$$

$$
\nabla_\nu \delta\Gamma^\rho_{\rho\mu} = \partial_\nu \delta\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\rho\mu} - \Gamma^\lambda_{\nu\rho}\delta\Gamma^\rho_{\lambda\mu} - \Gamma^\lambda_{\mu\nu}\delta\Gamma^\rho_{\rho\lambda} \tag{3.22}
$$

Therefore

$$\begin{aligned}\delta R_{\mu\nu} = &\partial_\rho \delta\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\rho\lambda}\delta\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\rho\mu}\delta\Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\rho\nu}\delta\Gamma^\rho_{\mu\lambda} + \Gamma^\lambda_{\rho\nu}\delta\Gamma^\rho_{\mu\lambda} + \\ &- \partial_\nu \delta\Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\rho\mu} + \Gamma^\lambda_{\nu\rho}\delta\Gamma^\rho_{\rho\mu} - \Gamma^\lambda_{\mu\nu}\delta\Gamma^\rho_{\rho\lambda} + \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\rho\mu} = \\ = &\nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}\end{aligned} \tag{3.23}$$

Then

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} &= \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\left(\nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}\right) = \\ &= \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\rho\left(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\nu_{\nu\mu}\right) = \\ &= \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\nabla_\rho J^\rho\right) = \\ &= \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}\left(n_\rho J^\rho\right)\end{aligned} \tag{3.24}$$

where

$$J^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\delta\Gamma^\beta_{\alpha\beta} \tag{3.25}$$

where $h_{\mu\nu}$ is the induced metric on the boundary as defined in (3.5) and n_μ is the normal vector to the boundary as defined before.
It is useful to prove the following property

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\gamma}\left(\nabla_\alpha \delta g_{\beta\gamma} + \nabla_\beta \delta g_{\alpha\gamma} - \nabla_\gamma \delta g_{\alpha\beta}\right) \tag{3.26}$$

In fact

$$\delta\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\delta g^{\mu\gamma}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right) + \frac{1}{2}g^{\mu\gamma}\left(\partial_\alpha \delta g_{\beta\gamma} + \partial_\beta \delta g_{\gamma\alpha} - \partial_\gamma \delta g_{\alpha\beta}\right) \tag{3.27}$$

Using the property [1]

$$\delta g^{\mu\gamma} = -g^{\mu\sigma}g^{\gamma\rho}\delta g_{\sigma\rho} \tag{3.28}$$

we can write

$$\begin{aligned}\delta\Gamma^\mu_{\alpha\beta} &= -\frac{1}{2}g^{\mu\sigma}g^{\gamma\rho}\delta g_{\sigma\rho}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right) + \frac{1}{2}g^{\mu\gamma}\left(\partial_\alpha \delta g_{\beta\gamma} + \partial_\beta \delta g_{\gamma\alpha} - \partial_\gamma \delta g_{\alpha\beta}\right) = \\ &= -g^{\mu\sigma}\Gamma^\rho_{\alpha\beta}\delta g_{\sigma\rho} + \frac{1}{2}g^{\mu\gamma}\left(\partial_\alpha \delta g_{\beta\gamma} + \partial_\beta \delta g_{\gamma\alpha} - \partial_\gamma \delta g_{\alpha\beta}\right)\end{aligned} \tag{3.29}$$

[1] It can be easily derived from

$$g_{\mu\nu}g^{\nu\gamma} = \delta g^\gamma_\mu \implies \delta\left(g_{\mu\nu}g^{\nu\gamma}\right) = 0$$

Then

$$\begin{aligned} g_{\mu\nu}\delta g^{\nu\gamma} &= -g^{\nu\gamma}\delta g_{\mu\nu} \\ g^{\mu\alpha}g_{\mu\nu}\delta g^{\nu\gamma} &= -g^{\mu\alpha}g^{\nu\gamma}\delta g_{\mu\nu} \\ \delta g^{\alpha\gamma} &= -g^{\mu\alpha}g^{\nu\gamma}\delta g_{\mu\nu}\end{aligned}$$

Renaming dummy indexes in the first term and adding and subtracting the same terms in order to reconstruct the covariant derivatives

$$\begin{aligned}\delta\Gamma^{\mu}_{\alpha\beta} &= \frac{1}{2}g^{\mu\gamma}\left(\partial_{\alpha}\delta g_{\beta\gamma} + \partial_{\beta}\delta g_{\gamma\alpha} - \partial_{\gamma}\delta g_{\alpha\beta} - 2\Gamma^{\rho}_{\alpha\beta}\delta g_{\gamma\rho}\right) = \\ &= \frac{1}{2}g^{\mu\gamma}\left(\nabla_{\alpha}\delta g_{\beta\gamma} + \nabla_{\beta}\delta g_{\alpha\gamma} - \nabla_{\gamma}\delta g_{\alpha\beta}\right)\end{aligned} \tag{3.30}$$

Now we can rewrite J^{μ} in a more convenient way

$$\begin{aligned}J^{\mu} &= \frac{1}{2}g^{\alpha\beta}g^{\mu\gamma}\left(\nabla_{\alpha}\delta g_{\beta\gamma} + \nabla_{\beta}\delta g_{\alpha\gamma} - \nabla_{\gamma}\delta g_{\alpha\beta}\right) + \\ &\quad - \frac{1}{2}g^{\alpha\mu}g^{\beta\gamma}\left(\nabla_{\alpha}\delta g_{\beta\gamma} + \nabla_{\beta}\delta g_{\alpha\gamma} - \nabla_{\gamma}\delta g_{\alpha\beta}\right) = \\ &= \frac{1}{2}g^{\alpha\beta}g^{\mu\gamma}\nabla_{\alpha}\delta g_{\beta\gamma} + \cancel{\frac{1}{2}g^{\gamma\mu}g^{\alpha\beta}\nabla_{\beta}\delta g_{\alpha\beta}} - \frac{1}{2}g^{\alpha\beta}g^{\mu\gamma}\nabla_{\gamma}\delta g_{\alpha\beta} + \\ &\quad - \frac{1}{2}g^{\gamma\beta}g^{\mu\alpha}\nabla_{\alpha}\delta g_{\beta\gamma} - \cancel{\frac{1}{2}g^{\alpha\mu}g^{\beta\gamma}\nabla_{\beta}\delta g_{\alpha\gamma}} + \frac{1}{2}g^{\alpha\mu}g^{\beta\gamma}\nabla_{\gamma}\delta g_{\alpha\beta} = \\ &= g^{\alpha\beta}g^{\mu\gamma}\nabla_{\alpha}\delta g_{\beta\gamma} - g^{\alpha\beta}g^{\mu\gamma}\nabla_{\gamma}\delta g_{\alpha\beta}\end{aligned} \tag{3.31}$$

3.3.2 Variation of the Gibbons-Hawking term

We have two terms coming from the variation of the Gibbons-Hawking term: $\delta\sqrt{h}$ and $\delta\Theta$.
The first one is easy to handle and it's equal to

$$\delta\sqrt{h} = \frac{1}{2}\sqrt{h}h^{\mu\nu}\delta h_{\mu\nu} \tag{3.32}$$

$\delta\Theta$ is the variation of the trace of the extrinsic curvature $\Theta_{\mu\nu}$

$$\begin{aligned}\Theta &= \Theta_{\mu\nu}g^{\mu\nu} = \\ &= -\frac{1}{2}\nabla_{\mu}n^{\mu} - \frac{1}{2}\nabla_{\nu}n^{\nu} = \\ &= -\nabla_{\mu}n^{\mu} = \\ &= -\left(\partial_{\mu}n^{\mu} + \Gamma^{\mu}_{\mu\lambda}n^{\lambda}\right)\end{aligned} \tag{3.33}$$

Its variation can be calculated as follows

$$\begin{aligned}\delta\Theta &= -\partial_{\mu}\delta n^{\mu} - \Gamma^{\mu}_{\mu\lambda}\delta n^{\lambda} - \delta\Gamma^{\mu}_{\mu\lambda}n^{\lambda} = \\ &= -\nabla_{\mu}\delta n^{\mu} - \delta\Gamma^{\mu}_{\mu\lambda}n^{\lambda} = \\ &= -\nabla_{\mu}\delta n^{\mu} - \frac{1}{2}g^{\mu\gamma}\left(\nabla_{\mu}\delta g_{\beta\gamma} + \nabla_{\beta}\delta g_{\mu\gamma} - \nabla_{\gamma}\delta g_{\mu\beta}\right)n^{\beta}\end{aligned} \tag{3.34}$$

3.3.3 Complete variation from the boundary terms

We now have all the terms to solve for the variation of the whole boundary term

$$
\begin{aligned}
\delta S_{\text{bdy}} &= \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}n_\mu J^\mu - \int_{\partial_0\mathcal{M}} d^3x\delta\sqrt{h}\Theta - \int_{\partial_0\mathcal{M}} d^3x\sqrt{h}\delta\Theta = \\
&= \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}n_\mu \underbrace{g^{\alpha\beta}g^{\mu\gamma}\left(\nabla_\alpha\delta g_{\beta\gamma} - \nabla_\gamma\delta g_{\alpha\beta}\right)}_{J^\mu} - \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}h^{\mu\nu}\Theta\delta h_{\mu\nu} + \\
&\quad + \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}\underbrace{\left(2\nabla_\mu\delta n^\mu + g^{\mu\gamma}\left(\nabla_\mu\delta g_{\beta\gamma} + \nabla_\beta\delta g_{\mu\gamma} - \nabla_\gamma\delta g_{\mu\beta}\right)n^\beta\right)}_{\delta\Theta} = \\
&= -\frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}h^{\mu\nu}\Theta\delta h_{\mu\nu} + \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}n^\gamma g^{\alpha\beta}\left(\nabla_\alpha\delta g_{\beta\gamma} - \nabla_\gamma\delta g_{\alpha\beta}\right) + \\
&\quad + \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}\left(g^{\mu\gamma}\left(\nabla_\mu\delta g_{\beta\gamma} + \nabla_\beta\delta g_{\mu\gamma} - \nabla_\gamma\delta g_{\mu\beta}\right)n^\beta\right) + \\
&\quad + \int_{\partial_0\mathcal{M}} d^3x\sqrt{h}\left(\nabla_\mu\delta n^\mu\right)
\end{aligned}
\tag{3.35}
$$

Since the last term is a total derivative on the border, we can neglect it. Instead we focus on the second and third terms

$$
\begin{aligned}
n_\mu J^\mu + \delta\Theta &= \cancel{n^\gamma g^{\alpha\beta}\nabla_\alpha\delta g_{\beta\gamma}} - \cancel{n^\gamma g^{\alpha\beta}\nabla_\gamma\delta g_{\alpha\beta}} + n^\beta g^{\mu\gamma}\nabla_\mu\delta g_{\beta\gamma} + \\
&\quad + \cancel{n^\beta g^{\mu\gamma}\nabla_\beta\delta g_{\mu\gamma}} - \cancel{n^\beta g^{\mu\gamma}\nabla_\gamma\delta g_{\mu\beta}} = \\
&= n^\beta g^{\mu\gamma}\nabla_\mu\delta g_{\beta\gamma} = \\
&= g^{\mu\gamma}\nabla_\mu\left(n^\beta\delta g_{\beta\gamma}\right) - g^{\mu\gamma}\nabla_\mu n^\beta\delta g_{\beta\gamma} = \\
&= g^{\mu\gamma}\nabla_\mu\left(\delta\left(n^\beta g_{\beta\gamma}\right) - \delta n^\beta g_{\beta\gamma}\right) - g^{\mu\gamma}\nabla_\mu n^\beta\delta g_{\beta\gamma} = \\
&= g^{\mu\gamma}\nabla_\mu\delta n_\gamma - g^{\mu\gamma}g_{\beta\gamma}\nabla_\mu\delta n^\beta - g^{\mu\gamma}\nabla_\mu n^\beta\left(-g_{\beta\rho}g_{\gamma\sigma}\delta g^{\rho\sigma}\right) = \\
&= \cancel{\nabla^\gamma\delta n_\gamma} - \cancel{\delta^\mu_\beta\nabla_\mu\delta n^\beta} + \delta^\mu_\sigma\nabla_\mu\left(n^\beta g_{\beta\rho}\right)\delta g^{\rho\sigma} = \\
&= \nabla_\sigma n_\rho\delta g^{\rho\sigma} = \\
&= -\nabla_\sigma n_\rho g^{\rho\alpha}g^{\sigma\beta}\delta g_{\alpha\beta} = \\
&= -\nabla^\alpha n^\beta\delta g_{\alpha\beta} = \\
&= \Theta^{\alpha\beta}\delta g_{\alpha\beta}
\end{aligned}
\tag{3.36}
$$

We observe that using the orthogonality relation $\Theta_{\mu\nu}n^\mu = 0 = \Theta_{\mu\nu}n^\nu$ and (3.5), we can write

$$
\begin{aligned}
\Theta^{\alpha\beta}\delta g_{\alpha\beta} &= \Theta^{\alpha\beta}\delta\left(h_{\alpha\beta} + n_\alpha n_\beta\right) = \\
&= \Theta^{\alpha\beta}\delta h_{\alpha\beta} + \underbrace{\Theta^{\alpha\beta}n_\alpha}_{0}\delta n_\beta + \underbrace{\Theta^{\alpha\beta}n_\beta}_{0}\delta n_\alpha = \\
&= \Theta^{\alpha\beta}\delta h_{\alpha\beta}
\end{aligned}
\tag{3.37}
$$

Finally the variation of the action on the boundary can be written as

$$\delta S_{\text{bdy}} = \frac{1}{2}\int_{\partial_0\mathcal{M}} d^3x\sqrt{h}\,(\Theta^{\mu\nu} - \Theta h^{\mu\nu})\,\delta h_{\mu\nu} \tag{3.38}$$

3.4 Regulated boundary stress-energy tensor

The quasilocal energy, (see [41], [42], [43]) is associated with the boundary stress-energy tensor, defined as

$$T^{\mu\nu} = \frac{2}{\sqrt{h}}\frac{\delta S_{\text{grav}}}{\delta h_{\mu\nu}} \tag{3.39}$$

Given the variation of the regulated on-shell action $\delta S_{\text{grav}} = \delta S_{\text{bdy}}$ calculated in (3.38) it is straightforward to derive the stress-energy tensor

$$T^{\mu\nu} = (\Theta^{\mu\nu} - \Theta h^{\mu\nu})\Big|_{r_0} \tag{3.40}$$

The mass is calculated from the tt component of $T_{\mu\nu}$ following the definition of a Komar charge, as in [2]. A Komar charge makes sense when we have a killing vector field, which in this case is ∂_t. Since time translations are related to energy conservation and given the general relativity mass-energy equivalence it makes sense that the Komar quantity associated with the time Killing vector represents the mass. Thus we integrate over a constant t surface on the boundary $\partial_0\mathcal{M}$

$$M_{reg} = Q_K = \frac{1}{8\pi}\int_{\Sigma} d^2x\sqrt{\sigma}u_\mu T^{\mu\nu}K_\nu \tag{3.41}$$

where Σ is a spacelike section of the boundary $\partial_0\mathcal{M}$ and σ is the metric induced on Σ. u_μ is the unit normal vector to Σ defined as

$$u^\mu = (\sqrt{h^{tt}}, 0, 0, 0); \qquad u_\mu = (\sqrt{h_{tt}}, 0, 0, 0) \tag{3.42}$$

and K^μ is the killing vector associated to time translations and thus to conserved energy

$$K^\mu = (1, 0, 0, 0); \qquad K_\mu = (h_{tt}, 0, 0, 0) \tag{3.43}$$

$$\sigma_{\mu\nu} = \begin{pmatrix} -e^{-K}r^2 & 0 \\ 0 & -e^{-K}r^2\sin^2\theta \end{pmatrix} \tag{3.44}$$

Then the regular part of the mass term is comuted as

$$M_{reg} = \frac{1}{8\pi} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \underbrace{\left(e^{-K} r^2 \sin\theta\right)}_{\sqrt{\sigma}} \underbrace{\left(e^{K/2}\sqrt{f}\right)}_{u_t} \underbrace{\left(\Theta^{tt} - h^{tt}\Theta\right)}_{T^{tt}} \underbrace{\left(e^{K} f\right)}_{K_t} \tag{3.45}$$

Θ is the trace of the extrinsic curvature as defined in (D.28) and Θ^{tt} is its tt component. They are given by [2]

$$\Theta = e^{K/2}\sqrt{f}\left(\frac{f'}{2f} + \frac{2}{r} - \frac{K'}{2}\right) \tag{3.46}$$

$$\Theta^{tt} = h^{tt} e^{K/2}\sqrt{f}\left(\frac{K'}{2} + \frac{f'}{2f}\right) \tag{3.47}$$

where K and f are as defined respectively in (2.20) and (2.14). Putting these values in (3.45) produces [3]

$$\begin{aligned} M_{reg} &= \frac{1}{8\pi} 4\pi r^2 f \left(K' - \frac{2}{r}\right) = \\ &= \frac{1}{2} r^2 f \left(\frac{-Q_1^2}{r(r-3Q_1)(r+Q_1)} - \frac{2}{r}\right) = \\ &= -rf\left(\frac{r^2 - 2rQ_1}{(r-3Q_1)(r+Q_1)}\right) = \\ &= -rf\left(1 + \frac{3Q_1^2}{r^2} + \frac{6Q_1^3}{r^3} + O\left(\frac{1}{r^4}\right)\right) = \\ &= -r - c_1 - \frac{r^3}{l_{AdS}^2} + \frac{8Q_1^3}{l_{AdS}^2} - \frac{3Q_1^2 r}{l_{AdS}^2} - \frac{6Q_1^3}{l_{AdS}^2} = \\ &= -r - c_1 - \frac{r^3}{l_{AdS}^2} + \frac{3Q_1^2 r}{l_{AdS}^2} + \frac{2Q_1^3}{l_{AdS}^2} \end{aligned} \tag{3.48}$$

Then we have the following value for the regular part of the mass term

$$M_{reg} = -r - c_1 - \frac{r^3}{l_{AdS}^2} + \frac{3Q_1^2 r}{l_{AdS}^2} + \frac{2Q_1^3}{l_{AdS}^2} \tag{3.49}$$

3.5 Counterterms and Finite Terms

In order to get rid of the linear and cubic divergencies appearing in M_{reg}, we should add a counterterm S_{ct} to the action S_{grav}.

[2]The explicit calculations are performed in the Appendix D

[3]Remember the expansion

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6}x^3 + ...$$

The contribution to the mass term is analogous to the computation in (3.41), using the counterterm stress-energy tensor $T^{\mu\nu}_{\text{ct}}$, defined as

$$T^{\mu\nu}_{\text{ct}} = \frac{2}{\sqrt{h}} \frac{\delta S_{\text{ct}}}{h_{\mu\nu}} \tag{3.50}$$

In addition to this we should impose specific boundary conditions on the scalar field as discussed in 2.3.2. The specific form of S_{ct} is different in the two work that we are comparing as well as the choice of boundary conditions.

3.5.1 Gnecchi-Toldo (GT) prescription

In [2] it is suggested to take the following form for S_{ct}

$$S_{\text{ct,GT}} = \int_{\partial\mathcal{M}_0} d^3x\sqrt{h}\left(-W(\varphi) + Z(\varphi)\mathcal{R}_{(3)}\right) \tag{3.51}$$

where $W(\varphi)$ is the superpotential

$$W(\varphi) = -\frac{2}{l_{AdS}}\left(1 + \frac{\varphi^2}{4}\right), \tag{3.52}$$

$Z(\varphi)$ is a function defined as

$$Z(\varphi) \sim -\frac{l_{AdS}}{2(d-2)}, \tag{3.53}$$

and $\mathcal{R}_{(3)}$ is the Ricci scalar defined on $\partial\mathcal{M}_0$ and explicitly calculated in (D.5).
Varying S_{ct} we obtain

$$\begin{aligned}
\delta S_{\text{ct,GT}} &= \int_{\partial\mathcal{M}_0} d^3x \left[\delta\sqrt{h}\left(-W(\varphi) + Z(\varphi)\mathcal{R}_{(3)}\right) + \sqrt{h}Z(\varphi)\left(\delta\mathcal{R}^{\mu\nu}_{(3)}h_{\mu\nu} + \mathcal{R}^{\mu\nu}_{(3)}\delta h_{\mu\nu}\right)\right] = \\
&= \int_{\partial\mathcal{M}_0} d^3x\frac{1}{2}\sqrt{h}h^{\mu\nu}\delta h_{\mu\nu}\left(-W(\varphi) + Z(\varphi)\mathcal{R}_{(3)}\right) + \int_{\partial\mathcal{M}_0} d^3x\sqrt{h}Z(\varphi)\mathcal{R}^{\mu\nu}_{(3)}\delta h_{\mu\nu} + \\
&\quad + \int_{\partial\mathcal{M}_0} d^3x\sqrt{h}Z(\varphi)h_{\mu\nu}\delta\mathcal{R}^{\mu\nu}_{(3)} = \\
&= \frac{1}{2}\int_{\partial\mathcal{M}_0} d^3x\sqrt{h}\left(-W(\varphi)h^{\mu\nu} + Z(\varphi)\left(h^{\mu\nu}\mathcal{R}_{(3)} + 2\mathcal{R}^{\mu\nu}_{(3)}\right)\right)\delta h_{\mu\nu}
\end{aligned} \tag{3.54}$$

since the last term is a total divergence on the boundary $\partial_0\mathcal{M}$ and does not contribute.
Then

$$T^{\mu\nu}_{\text{ct,GT}} = -W(\varphi)h^{\mu\nu} + Z(\varphi)\left(h^{\mu\nu}\mathcal{R}_{(3)} + 2\mathcal{R}^{\mu\nu}_{(3)}\right) \tag{3.55}$$

Now we can calculate the counterterm contribution to the mass

$$
\begin{aligned}
M_{\text{ct,GT}} &= \frac{1}{8\pi}\int_{\Sigma} d^2x\sqrt{\sigma}u_\mu T^{\mu\nu}_{\text{ct}}K_\nu = \\
&= \frac{1}{8\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \underbrace{\left(e^{-K}r^2\sin\theta\right)}_{\sqrt{\sigma}} \underbrace{\left(e^{K/2}\sqrt{f}\right)}_{u_t} \underbrace{\left(-W(\varphi)h^{tt}+Z(\varphi)h^{tt}\mathcal{R}_{(3)}\right)}_{T^{tt}_{\text{ct}}} \underbrace{\left(e^{K}f\right)}_{K_t} = \\
&= \underbrace{-\frac{1}{2}e^{-K/2}r^2\sqrt{f}W(\varphi)}_{I_1} + \underbrace{\frac{1}{2}e^{-K/2}r^2\sqrt{f}Z(\varphi)\mathcal{R}_{(3)}}_{I_2}
\end{aligned} \tag{3.56}
$$

where we have used the fact that $\mathcal{R}^{tt}_{(3)} = 0$.
Substituting the values for φ from (2.81), $W(\varphi)$ from (3.52), $Z(\varphi)$ from (3.53) and $\mathcal{R}_{(3)}$ from (D.5), we obtain

$$
\begin{aligned}
I_1 &= -\frac{1}{2}e^{-K/2}r^2\sqrt{f}W(\varphi) = \\
&= \frac{1}{2}\left(\frac{(r+Q_1)^3(r-3Q_1)}{r^4}\right)^{1/4} r^2\frac{r}{l_{AdS}}\left(1+\frac{l_{AdS}^2}{2r^2}+\frac{c_1l_{AdS}^2}{2r^3}+\frac{c_2l_{AdS}^2}{2r^4}-\frac{3Q_1^2}{r^2}-\frac{4Q_1^3}{r^3}-\frac{3Q_1^4}{r^4}\right)\cdot \\
&\quad\cdot\left(\frac{2}{l_{AdS}}\right)\left(1+\frac{3Q_1^2}{2r^2}+\frac{3Q_1^3}{r^3}+\frac{3}{2}\frac{Q_1^4}{r^4}\right) = \\
&= \frac{r^2}{l_{AdS}^2}r\left(1-\frac{6}{4}\frac{Q_1^2}{r^2}-\frac{8}{4}\frac{Q_1^3}{r^3}-\frac{3}{4}\frac{Q_1^4}{r^4}\right)\left(1+\frac{l_{AdS}^2}{2r^2}+\frac{c_1l_{AdS}^2}{2r^3}+\frac{c_2l_{AdS}^2}{2r^4}-\frac{3Q_1^2}{r^2}-\frac{4Q_1^3}{r^3}-\frac{3Q_1^4}{r^4}\right)\cdot \\
&\quad\cdot\left(1+\frac{3Q_1^2}{2r^2}+\frac{3Q_1^3}{r^3}+\frac{3}{2}\frac{Q_1^4}{r^4}\right) = \\
&= \frac{r^3}{l_{AdS}^2}+\frac{r}{2}+\frac{c_1}{2}-\frac{3Q_1^2r}{l_{AdS}^2}-\frac{3Q_1^3}{l_{AdS}^2}
\end{aligned} \tag{3.57}
$$

$$
\begin{aligned}
I_2 &= \frac{1}{2}e^{-K/2}r^2\sqrt{f}Z(\varphi)\mathcal{R}_{(3)} = \\
&= \frac{1}{2}e^{-K/2}r^2\frac{r}{l_{AdS}}\left(1+\frac{l_{AdS}^2}{2r^2}+\frac{c_1l_{AdS}^2}{2r^3}+\frac{c_2l_{AdS}^2}{2r^4}-\frac{3Q_1^2}{r^2}-\frac{4Q_1^3}{r^3}-\frac{3Q_1^4}{r^4}\right)\left(-\frac{l_{AdS}}{2}\right)\left(\frac{2e^K}{r^2}\right) = \\
&= \frac{r}{2}\left(\frac{r^4}{r^4-6r^2Q_1^2-8rQ_1^3-3Q_1^4}\right)^{1/4}\left(1+\frac{l_{AdS}^2}{2r^2}+\frac{c_1l_{AdS}^2}{2r^3}+\frac{c_2l_{AdS}^2}{2r^4}-\frac{3Q_1^2}{r^2}-\frac{4Q_1^3}{r^3}-\frac{3Q_1^4}{r^4}\right) = \\
&= \frac{r}{2}
\end{aligned} \tag{3.58}
$$

So

$$
M_{\text{ct,GT}} = r+\frac{c_1}{2}+\frac{r^3}{l_{AdS}^2}-\frac{3Q_1^2r}{l_{AdS}^2}-\frac{3Q_1^3}{l_{AdS}^2} \tag{3.59}
$$

Mixed boundary conditions are imposed on the scalar field as defined in (2.88). This shifts the mass by a finite term given by

$$\begin{aligned}M_{\text{fin,GT}} &= \frac{1}{8\pi}\int_{\partial\mathcal{M}_0}\sqrt{h_{(0)}}h^{(0)}_{tt}\tilde{f}(\varphi_-) = \\ &= \frac{1}{8\pi}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\underbrace{(l^2_{AdS}\sin\theta)}_{\sqrt{h_{(0)}}}\underbrace{(1)}_{h^0_{tt}}\underbrace{\left(2\frac{Q_1^3}{l^4_{AdS}}\right)}_{\tilde{f}(\varphi_-)}\end{aligned}\tag{3.60}$$

$$M_{\text{fin,GT}} = \frac{Q_1^3}{l^2_{AdS}}\tag{3.61}$$

Therefore

$$M_{GT} = -\frac{c_1}{2}\tag{3.62}$$

3.5.2 Halmagyi-Lal (HL) prescription

In [1] it is suggested to use only a part of the superpotential[4] W written in terms of symplectic invariants as follows

$$S_{\text{ct,HL}} = \int_{\partial\mathcal{M}_0}d^3x\sqrt{h}\left(2\text{Im}\left(e^{i\alpha}\mathcal{L}\right)+Z(\varphi)\mathcal{R}_{(3)}\right)\tag{3.64}$$

where $\mathcal{L}$ is given in (A.37), Z is given in (3.53) and $\mathcal{R}_{(3)}$ is calculated in (D.5).
The contribution to the mass term is analogue to (3.56). The I_1 part only changes as follows

$$\begin{aligned}I_{1,HL} = &= \frac{1}{2}e^{-K/2}r^2\sqrt{f}\,(2\mathcal{L}) = \\ &= r^2\left(\frac{r}{l^2_{AdS}}\left(1+\frac{1}{2}\frac{l^2_{AdS}}{r^2}+\frac{1}{2}\frac{c_1l^2_{AdS}}{r^3}+c_2\frac{l^2_{AdS}}{2r^4}-3\frac{Q_1^2}{r^2}-4\frac{Q_1^3}{r^3}-3\frac{Q_1^4}{2r^4}\right)\right) = \\ &= \frac{r^3}{l^2_{AdS}}+\frac{r}{2}+\frac{c_1}{2}-\frac{3Q_1^2r}{l^2_{AdS}}-\frac{4Q_1^3}{l^2_{AdS}}\end{aligned}\tag{3.65}$$

Then

$$M_{\text{ct,HL}} = r+\frac{c_1}{2}+\frac{r^3}{l^2_{AdS}}-\frac{3Q_1^2r}{l^2_{AdS}}-\frac{4Q_1^3}{l^2_{AdS}}\tag{3.66}$$

[4]Here the superpotential is written as

$$W = h^2U\text{Im}\left(e^{-i\alpha}\mathcal{L}\right)+U\text{Re}\left(e^{-i\alpha}\mathcal{Z}\right)\tag{3.63}$$

where h and U are the metric warp factor as defined respectively in (2.8) and (2.7), α is the supersymmetry phase factor as defined in (B.8) and $\mathcal{L}$ and $\mathcal{Z}$ are special geometry quantities defined respectively in (A.37) and (A.49).

Halmagyi suggests Neumann boundary condition for the scalar field as defined in (2.87). Thus the finite shift in this case is given by

$$M_{\text{fin,HL}} = \frac{6Q_1^3}{l_{AdS}^2} \tag{3.67}$$

where we have used (2.90).
Finally we have the following value for the mass following the prescription in [1]

$$M_{HL} = -\frac{c_1}{2} + \frac{4Q_1^3}{l_{AdS}^2} \tag{3.68}$$

4 Entropy and Extremization Principle

In this chapter we are going to address the main objective of this thesis. We are going to compare [1] and [2] with respect to the validity of

$$S_{\text{on-shell}}\Big|_{BPS} = -S \tag{4.1}$$

(4.1) serves for consistency in the context of microscopic counting for the entropy of AdS black holes and the so-called extremization principle.

Microscopic counting and extremization principle

Considering a generic black hole in AdS, we have that microscopic counting is done on the field theory side. Extremal supersymmetric AdS black holes are considered and their entropy is calculated by counting the BPS states of the dual field theory, having the same amount of supersymmetry as the black hole and living on the conformal boundary of AdS (see review [15]). Counting the BPS states of the CFT results in evaluating its grancanonical partition function $\mathcal{Z}$

$$\mathcal{Z}(\Delta_i, \omega_j) = \sum_{q_i, j_i} c(q_i, j_j) e^{i(\Delta_i q_i + \omega_j j_j)} \tag{4.2}$$

where q_i are the electric charges, j_j the angular momenta, while Δ_i and ω_j are their respective conjugated chemical potentials. $c(q_i, j_j)$ represents the number of states of given q_i and j_j. Thus the entropy of the AdS black hole is given by

$$S(q_i, j_j) = \log c(q_i, j_j) \tag{4.3}$$

$S(q_i, j_j)$ can be calculated from (4.2) as a Fourier coefficient. In the limit of large charges (large N) the computation can be reduced to a saddle point approximation

$$S(q_i, j_j) = \left[\log \mathcal{Z}(\Delta_i, \omega_j) - i(\Delta_i q_i + \omega_j j_j)\right] |_{\bar{\Delta}_i, \bar{\omega}_j} \tag{4.4}$$

where $\bar{\Delta}_i, \bar{\omega}_j$ are the values obtained extremizing the functional on the right hand side of (4.4). This is what is referred to as the extremization principle.
On the other hand, we have that the holographic dictionary relates the dual quantum field theory partition function to the gravitational on-shell action $S_{\text{on-shell}}$, when approximating the last one to the free energy

$$\mathcal{Z}_{\text{CFT}} = e^{-\beta\Omega} \sim e^{-S_{\text{on-shell}}} \tag{4.5}$$

where Ω is the free energy.

Considering static and magnetically charged black holes, we have analyzed in this work, we have the thermodynamic relation between the black hole entropy and the field theory partition function on one side, and the holographic equivalence of the partition function and the on-shell gravitational action, on the other. It follows that the on-shell gravitational action should, in turn, reproduce (minus) the black hole entropy. The equivalences can be summarized in the following

$$S_{\text{on-shell}} = -\log Z_{\text{CFT}} = -S \tag{4.6}$$

The clarification of these relations was the starting point for the work in [1]. Here we are going to analyze [1], which states the validity of (4.1), using the solution in [2].
In the first part we directly compute the on-shell gravitational action by means of the one-dimensional reduction procedure. We are checking both the correspondence between the value of the on-shell action and the free energy (which for the case in exam is the Helmholtz free energy ([2])

$$S_{\text{on-shell}} = \beta\Omega \tag{4.7}$$

and (4.1).
We conclude with a brief remark on BPS bounds for the mass of magnetic black holes and some observations regarding the first law of thermodynamics.

4.1 The on-shell action

We shall now directly evaluate the on-shell action. Remember that we are considering

$$8\pi G = 1 \tag{4.8}$$

To do so we will write the one-dimensional effective action, ([1], [35]), using the relations (C.16), (C.17) and (C.18), which we report here for convenience:

$$g_{zz}\partial_r z \partial_r z = -\frac{h''}{h} \tag{4.9}$$

$$V_g = \frac{1}{2h^2}\left(1 - \frac{1}{2}(U^2h^2)''\right) \tag{4.10}$$

$$V_{BH} = \frac{1}{2}h^2\left(1 - U^2h'^2 - hh''U^2 + h^2U'^2 + h^2UU''\right) \tag{4.11}$$

From now on we are going to deal with the euclidean action related to the Minkowskian by a minus sign

$$S_{\text{euclidean}} = -S_{\text{Minkowski}} \tag{4.12}$$

The euclidean action is periodic in time with period β.
Therefore the dimensionally reduced action becomes

$$\begin{aligned}
S_{\text{bulk}} &= -\beta\int d^3x\sqrt{-g}\left(\frac{R}{2} + g_{zz}\partial_\mu z\partial^\mu z + \mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} - V_g\right) = \\
&= -\beta\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\sin\theta\int_{r_+}^{r_0}h^2\left[\frac{1}{h^2}\left(1 - h^2U'^2 - U^2(h'^2 + 2hh'')+\right.\right. \\
&\qquad\left.\left. - hU(4h'U' + hU'')\right) + \frac{h''}{h}U^2 - \frac{V_{BH}}{h^4} - \frac{1}{2h^2}\left(1 - \frac{1}{2}(U^2h^2)''\right)\right] = \\
&= (-4\pi\beta)\int_{r_+}^{r_0}dr\left[-\left(h^2U'^2 + h^2UU'' + 2hh'UU'\right)\right] = \\
&= (4\pi\beta)\int_{r_+}^{r_0}dr\left[\frac{1}{2}\left(h^2(U^2)'\right)'\right]
\end{aligned} \tag{4.13}$$

where r_0 is the radial cut-off as explained in chapter 3.
The bulk contribution is a total derivative term:

$$S_{\text{bulk}} = (4\pi\beta)\left[h^2UU'\Big|_{r_0} - h^2UU'\Big|_{r_+}\right] \tag{4.14}$$

We have both a boundary and a horizon terms. Let's evaluate the horizon term:

$$h^2UU'\Big|_{r_+} = 2ST \tag{4.15}$$

where S is the entropy as in (2.59) and T is the temperature as in (2.55).
Before evaluating the boundary contribution, we should first combine it with the

Gibbons-Hawking term. [1]

$$S_{\text{GHY}} = -\beta \int_{r_0} d^2x \sqrt{h}\Theta = \\ = (4\pi\beta)\left[-2hh'U^2 - h^2UU'\right]\Big|_{r_0} \tag{4.16}$$

We note that the h^2UU' term cancels and we are left with

$$\begin{aligned} S_{r_\infty} &= (8\pi\beta)\left[-hh'U^2\right]_{r_0} = \\ &= (8\pi\beta)\left[\left(e^{-K/2}r\right)\left(e^{-K/2}\left(-1+\frac{K'}{2}r\right)\right)\left(e^K f\right)\right]_{r_0} = \\ &= (8\pi\beta)\left[-rf + \frac{r^2}{2}fK'\right]_{r_0} = \\ &= (8\pi\beta)\left[-r - c_1 - \frac{r^3}{l_{AdS}^2} + \frac{3Q_1^2 r}{l_{AdS}^2} + \frac{2Q_1^3}{l_{AdS}^2}\right]_{r_0} = \\ &= 8\pi\beta\left[-r - c_1 - \frac{r^3}{l_{AdS}^2} + \frac{3Q_1^2 r}{l_{AdS}^2} + \frac{2Q_1^3}{l_{AdS}^2}\right]_{r_0} \end{aligned} \tag{4.17}$$

The boundary term shows linear and cubic divergencies. We shall get rid of them by adding appropriate counterms. Let's analyze two proposals, the one in [1] and the one in [2], in analogy with what already done in chapter 3.

4.1.1 Gnecchi-Toldo (GT) prescription

Now we analyze the proposal as in [2]. We report here the counterterm action

$$S_{\text{ct,GT}} = \int_{r_0} d^3x \sqrt{h}\left[-W(\varphi) + Z\mathcal{R}_{(3)}\right] \tag{4.18}$$

[1] In the following we show how to write the $\sqrt{h}\Theta$ term using the metric warp factors. Θ is as derived in (D.28) and

$$\sqrt{h} = r^2\sqrt{f}e^{-K/2}\sin\theta$$

So we have

$$\sqrt{h}\Theta = \sin\theta r^2 f\left(\frac{f'}{2f} + \frac{2}{r} - \frac{K'}{2}\right)$$

Remembering that the warp factors of the metric are given by $h = e^{-K/2}r$ and $U = e^{K/2}\sqrt{f}$, notice that

$$2Uh' + U'h = \sqrt{f}r\left(\frac{f'}{2f} + \frac{2}{r} - \frac{K'}{2}\right)$$

Then

$$\sqrt{h}\Theta = \sin\theta\left(2U^2hh' + h^2UU'\right)$$

We use the same values for Z and $\mathcal{R}_{(3)}$ as in Chapter 3, and $W(\varphi)$ is as in (3.52):

$$\begin{aligned} S_{\text{ct,T}} &= (4\pi\beta)\left[\frac{2}{l_{AdS}}e^{-K/2}r^2\sqrt{f}\left(1+\frac{\varphi^2}{4}\right)+e^{K/2}l_{AdS}\sqrt{f}\right]_{r_0} = \\ &= (8\pi\beta)\left[\frac{r^3}{l_{AdS}^2}+\frac{c_1}{2}-\frac{3Q_1^2 r}{l_{AdS}^2}+r-\frac{3Q_1^3}{l_{AdS}^2}\right]_{r_0} \end{aligned} \tag{4.19}$$

In this case, for the total action we obtain

$$\begin{aligned} S_{\text{on-shell}} &= S_{\text{bulk}}+S_{r_\infty}+S_{\text{ct,T}} = \\ &= \beta\left[-\frac{S}{\beta}-\frac{c_1}{2}-\frac{Q_1^3}{l_{AdS}^2}\right] \end{aligned} \tag{4.20}$$

Adding a finite term for mixed boundary conditions (see equations (2.88) and (3.61)), we get

$$S_{\text{on-shell,GT}} = \beta\left(-\frac{S}{\beta}-\frac{c_1}{2}\right) \tag{4.21}$$

which satisfies the holographic relation between temperature and free energy, but does not satisfies (4.1), since the mass term does not vanish in the limit.

$$S_{\text{on-shell,GT}}\bigg|_{BPS} = \lim_{\substack{T\to 0\\ c_1\to \frac{8Q_1^3}{l_{AdS}^2}}}\left(-S-4\beta\frac{Q_1^3}{l_{AdS}^2}\right) \tag{4.22}$$

4.1.2 Halmagyi-Lal (HL) prescription

We now analyze the counterterms proposed in [1], which we report here for convenience

$$S_{\text{ct,H}} = \int_{r_0} d^3x\sqrt{h}\left[2\text{Im}\left(e^{-i\alpha}\mathcal{L}\right)+Z\mathcal{R}_{(3)}\right] \tag{4.23}$$

Using (B.8), (B.33), (D.5) and (3.53), we obtain

$$\begin{aligned} S_{\text{ct,H}} &= (-\beta)(4\pi)\left[-\frac{2Uh^2e^{K/2}}{l_{AdS}}-\frac{2l_{AdS}Uh^2}{2h^2}\right]_{r_0} = \\ &= (4\pi\beta)\left[\frac{2r^2}{l_{AdS}}\sqrt{f}+l_{AdS}e^{K/2}\sqrt{f}\right]_{r_0} = \\ &= (4\pi\beta)\left[\frac{2r^3}{l_{AdS}^2}+c_1-\frac{8Q_1^3}{l_{AdS}^2}+2r-\frac{6Q_1^2 r}{l_{AdS}^2}\right]_{r_0} = \\ &= 8\pi\beta\left[\frac{r^3}{l_{AdS}^2}+\frac{c_1}{2}-\frac{4Q_1^3}{l_{AdS}^2}+r-\frac{3Q_1^2 r}{l_{AdS}^2}\right]_{r_0} \end{aligned} \tag{4.24}$$

Summing up all the contributions and taking the $r_0 \to \infty$ limit, we get

$$S_{\text{on-shell}} = \beta\left(-\frac{S}{\beta}-\frac{c_1}{2}-\frac{2Q_1^3}{l_{AdS}^2}\right) \tag{4.25}$$

where we have set $8\pi = 1$.
Adding the finite term coming from imposing Neumann boundary conditions (see equations (2.87) and (3.67)) we end up with

$$S_{\text{on-shell,HL}} = \beta \left(-\frac{S}{\beta} - \frac{c_1}{2} + \frac{4Q_1^3}{l_{AdS}^2} \right) \tag{4.26}$$

Notice that the holographic relation between the on-shell action and the free energy $\Omega = M - TS$

$$S_{\text{on-shell}} = \beta\Omega \tag{4.27}$$

is satisfied.
Moreover taking the extremal limit we do recover the extremization principle as the mass term goes to zero

$$M_{HL}|_{BPS} = \lim_{c_1 \to \frac{8Q_1^3}{l_{AdS}^2}} \left(-\frac{c_1}{2} + 4\frac{Q_1^3}{l_{AdS}^2} \right) = 0 \tag{4.28}$$

Then (4.1) holds

$$S_{\text{on-shell,HL}}\Big|_{BPS} = -S \tag{4.29}$$

Some observations on thermodynamics

We are now going to analyze the validity of the first law of thermodynamics which for magnetic black holes can be restated as

$$dM = TdS + \chi_\Lambda dp^\Lambda \tag{4.30}$$

We are going to consider the two different mass values, M_{GT} and M_{HL} as respectively given in (3.62) and (3.68). We refer to chapter 2 for the values of magnetic charges p^Λ, magnetostatic potentials ξ_Λ, temperature T and entropy S.
With these quantities at hand we have that

- Fot the Gnecchi-Toldo mass the first law of thermodynamics is valid and continues to hold in the extremal limit. We report here the quantities comparing in (4.30) and their respective extremal limits

$$M_{GT} = -\frac{c_1}{2} \quad \xrightarrow[BPS]{} \quad -\frac{4Q_1^3}{l_{AdS}^2} \tag{4.31}$$

$$T = \frac{1}{4\pi} e^{K(r)} \frac{df(r)}{dr}\bigg|_{r=r_+} \quad \xrightarrow[BPS]{} \quad \frac{1}{4\pi} e^{K(r)} \frac{df(r)}{dr}\bigg|_{r=r_h} = 0 \tag{4.32}$$

$$S = S = \pi r_+^2 e^{-K(r_+)} \quad \xrightarrow[BPS]{} \quad S = \pi r_h^2 e^{-K(r_h)} \tag{4.33}$$

where

$$r_h = \sqrt{3Q_1^2 - \frac{l_{AdS}^2}{2}} \tag{4.34}$$

At last we give the expressions for the magnetic charges and the magnetostatic potentials, which agree with those of [2]

$$p^0 = \pm \frac{1}{2 l_{AdS} g_0} \sqrt{c_2 + 3Q_1(3Q_1 + c_1)} \quad \xrightarrow[BPS]{} \quad -\frac{1}{4g_0} - \frac{3Q_1^2}{g_0 l_{AdS}^2} \tag{4.35}$$

$$p^1 = \pm \frac{3}{2 l_{AdS} g_1} \sqrt{c_2 + Q_1(Q_1 - c_1)} \quad \xrightarrow[BPS]{} \quad -\frac{3}{4g_1} + \frac{3Q_1^2}{g_1 l_{AdS}^2} \tag{4.36}$$

$$\chi_0 = g_0^2 l_{AdS}^2 \frac{p^0}{r_+ - 3Q_1} \quad \xrightarrow[BPS]{} \quad g_0^2 l_{AdS}^2 \frac{p^0}{r_h - 3Q_1} \tag{4.37}$$

$$\chi_1 = g_1^2 l_{AdS}^2 \frac{p^1}{3(Q_1 + r_+)} \quad \xrightarrow[BPS]{} \quad g_1^2 l_{AdS}^2 \frac{p^1}{3(Q_1 + r_h)} \tag{4.38}$$

where c_2 satisfies

$$c_2 = -(c_1 + r_+) r_+ - \frac{1}{l_{AdS}^2} (r_+ - 3Q_1)(Q_1 + r_+)^3 \tag{4.39}$$

In the BPS limit we have only one independent variable Q_1, since c_1 becomes a function of Q_1, while at finite temperature we should variate each quantity with respect to both c_1, Q_1. We obtain

– Varying with respect to c_1

$$\frac{dM_{GT}}{dc_1} = -\frac{1}{2}; \quad T\frac{dS}{dc_1} = 0; \quad \chi_0 \frac{dp^0}{dc_1} + \chi_1 \frac{dp^1}{dc_1} = -\frac{1}{2} \tag{4.40}$$

– Varying with respect to Q_1

$$\begin{gathered}\frac{dM_{GT}}{dQ_1} = 0; \quad T\frac{dS}{dQ_1} = \frac{-3Q_1\left(c_1 l_{AdS}^2 - 8Q_1^3 + 2l_{AdS}^2 r_+ - 12Q_1^2 r_+ + 4r_+^3\right)}{2l_{AdS}^2 (r_+ - 3Q_1)(r_+ + Q_1)} \\ \chi_0 \frac{dp^0}{dQ_1} + \chi_1 \frac{dp^1}{dQ_1} = \frac{3Q_1\left(c_1 l_{AdS}^2 - 8Q_1^3 + 2l_{AdS}^2 r_+ - 12Q_1^2 r_+ + 4r_+^3\right)}{2l_{AdS}^2 (r_+ - 3Q_1)(r_+ + Q_1)}\end{gathered} \tag{4.41}$$

All the differentials cancel exactly and so the first law of thermodynamics holds for any T.

- If we consider the mass obtained with the Halmagyi-Lal procedure, while maintaining all the other quantities as in Toldo, then

$$M_{HL} = -\frac{c_1}{2} + \frac{4Q_1^3}{l_{AdS}^2} \xrightarrow[BPS]{} 0 \tag{4.42}$$

So

$$dM_{HL} = -\frac{1}{2}dc_1 + 12\frac{Q_1^2}{l_{AdS}^2}dQ_1 \tag{4.43}$$

Thus we have that this does not satisfy the first law of thermodynamics.

We shall analyze this considering the works in [21] and [22]. Here BPS bounds for the mass of black holes in anti-de Sitter spacetime are analyzed. They are based on the previous results of [40] and [18], but put a special attention when the magnetic charges are present, as is our case.
It is shown that when black holes are magnetically charged the BPS bound for the mass is given by

$$M \geq 0 \tag{4.44}$$

Indeed the Halmagyi-Lal mass saturates this BPS bound and this is the reason why their solution satisfies the relation (4.1) in the first place.
The Gnecchi-Toldo mass, instead, even when evaluated at the extremal limit lies above the BPS bound and this prevents the validity of (4.1), but satisfies the first law of thermodynamics.

The fact that the mass saturating the BPS limit was not the mass satisfying the first law of thermodynamics was already noted, as a marginal note, in [44].
The reason might reside in the way the magnetostatic potentials, χ_Λ are computed (see (2.48)).

5 Conclusions

In this thesis we dealt with static, magnetically charged AdS_4 black holes with a spherical horizon, arising from $\mathcal{N} = 2$ Fayet-Ilioupoulos supergravity, in order to analyze their thermodynamic properties.

We gave a brief introduction on black hole thermodynamics, described the main characteristics of $\mathcal{N} = 2$ gauged supergravity and of the class of magnetic black holes we were considering. In particular, we focused on the holographic renormalization technique to compute the black hole mass and the on-shell action, as they were fundamental for testing the relation between the on-shell gravitational action and the black hole entropy in the BPS limit.

The main theme was the comparison of the works by Gnecchi and Toldo, [2], and the one by Halmagyi and Lal, [1]. The scope was to resolve the apparent contradiction in the validity of

$$S_{\text{on-shell}}\Big|_{BPS} = -S \tag{5.1}$$

which should follow for consistency, when considering the thermodynamic relation between the black hole entropy and the dual field theory partition function, on one side, and the holographic correspondence between the dual field theory partition function and the on-shell gravitational action, on the other.

In [1] it is stated that (5.1) holds for generic black holes in AdS_4 coupled to running scalars, once the on-shell action has been properly renormalized by means of the holographic renormalization technique. Nevertheless, when testing the solution in [2], which describes the particular class of magnetic AdS_4 black holes with running scalars, we have analyzed here, the relation (5.1) does not hold.
The reason is to be found in both the choice of right counterterms to get rid of the infrared divergencies and in the choice of the boundary conditions for the scalar field. This last one has a mass in that allows for both Neumann and mixed boundary conditions.

We have found that the counterterms and boundary conditions (Neumann) adopted in [1], lead to a mass saturating the BPS bound for the mass of magnetic black holes and thus (5.1) is satisfied. The counterterms and boundary conditions (mixed) used in [2], produce, instead, a mass term that in the extremal limit lies above the BPS bound and does not satisfy the relation (5.1) for the entropy.

A Special Kähler Geometry

The scalars z^i in the vector multiplets of $\mathcal{N} = 2$ supergravity theories in $d = 4$ dimensions exhibit special Kähler geometry as they parametrize a special Kähler manifold.
The general form of kinetic terms of the vector fields is given by

$$\mathcal{L}_{\text{kin}} = (\text{Im}\mathcal{N}_{\Lambda\Sigma})\, F^{\Lambda}_{\mu\nu} F^{\Sigma\mu\nu} + \frac{1}{2}\,(\text{Re}\mathcal{N}_{\Lambda\Sigma})\, \epsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Sigma}_{\rho\sigma} \tag{A.1}$$

where the symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$, called period matrix, depends on the scalar fields and $F^{\Lambda}_{\mu\nu}$ are the field strengths of the vector fields. The imaginary part of the matrix $\mathcal{N}$ should be negative definite in order to provide the right sign for the kinetic term.
The period matrix is a link between the scalar and the vector sectors of the theory. The duality transformations, which regard the spin-1 fields field strengths and which take the form of symplectic transformations, impose constraints on the scalar fields which are central to the special geometry properties.

A.1 Symplectic Structure of a Special Kähler Manifold

A special Kähler manifold is a n_V dimesional Hodge-Kähler manifold (where n_V is the number of vector multiplets coupled to the gravity multiplet), which is the base of a symplectic bundle described by covariantly holomorphic sections, X^{Λ} and F_{Λ}. The sections are holomorphic functions of the scalar fields z^i.

$$\mathcal{V} = \begin{pmatrix} L^{\Lambda} \\ M_{\Lambda} \end{pmatrix} = e^{K/2} \begin{pmatrix} X^{\Lambda} \\ F_{\Lambda} \end{pmatrix} \tag{A.2}$$

$K = K(z^i, \bar{z}^{\bar{i}})$ is the Kähler potential and it is defined in terms of the sections as

$$K = -\log[i(\bar{X}^{\Lambda} F_{\Lambda} - X^{\Lambda} \bar{F}_{\Lambda})] \tag{A.3}$$

The corresponding Kähler metric is

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K \tag{A.4}$$

where $i, \bar{j}$ denote, respectively, the derivative with respect to z^i and $\bar{z}^{\bar{j}}$, which in turn is the complex conjugate of z^j.
Under certain conditions there exists a second degree homogeneous function $F(X)$, called the prepotential, such that

$$F_\Lambda = \partial_\Lambda F \tag{A.5}$$

For the model adopted in this work the prepotential is given by

$$F(X) = -2i\sqrt{X^0(X^1)^3} \tag{A.6}$$

When there is a prepotential the period matrix can be derived from the following relation

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i\frac{\mathrm{Im}(F_{\Lambda\Delta})X^\Delta \mathrm{Im}(F_{\Sigma\Phi})X^\Phi}{X^\Delta \mathrm{Im}(F_{\Delta\Phi})X^\Phi} \tag{A.7}$$

with

$$F_{\Lambda\Sigma} = \partial_\Lambda \partial_\Sigma F \tag{A.8}$$

Otherwise it is implicitly defined by the following matrix relations

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma, \qquad \bar{h}_{\Lambda,\bar{i}} = \mathcal{N}_{\Lambda\Sigma} \bar{f}^\Sigma_{\bar{i}} \tag{A.9}$$

$$\mathcal{N} = \begin{pmatrix} \bar{h}_{\Lambda,\bar{i}} \\ M_\Lambda \end{pmatrix} \cdot \begin{pmatrix} \bar{f}^\Sigma_{\bar{i}} \\ L^\Sigma \end{pmatrix}^{-1} \tag{A.10}$$

with

$$U_i \equiv D_i \mathcal{V} \equiv \left(\partial_i + \frac{1}{2}\partial_i K\right) \mathcal{V} \equiv \begin{pmatrix} f^\Lambda_i \\ h_{i,\Lambda} \end{pmatrix} \tag{A.11}$$

$$\bar{U}_{\bar{i}} \equiv D_{\bar{i}} \bar{\mathcal{V}} \equiv \left(\partial_{\bar{i}} + \frac{1}{2}\partial_{\bar{i}} K\right) \bar{\mathcal{V}} \equiv \begin{pmatrix} \bar{f}^\Lambda_{\bar{i}} \\ \bar{h}_{\bar{i},\Lambda} \end{pmatrix} \tag{A.12}$$

Using these notations we can write the scalar potential that appears in (2.5) as

$$V_g = (g^{i\bar{j}} f^\Lambda_i \bar{f}^\Sigma_{\bar{j}} - 3\bar{L}^\Lambda L^\Sigma) g_\Lambda g_\Sigma \tag{A.13}$$

A.1.1 Special Kähler Manifold Properties

Denoting the symplectic scalar product as

$$\langle A, B \rangle = A^T \Omega B = A_\Lambda B^\Lambda - A^\Lambda B_\Lambda \tag{A.14}$$

with

$$\Omega = \begin{pmatrix} \mathbb{0} & -\mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{pmatrix} \tag{A.15}$$

a set of useful properties can be derived.
The Kähler potential can be written as

$$K = -\log[i\langle \mathcal{V}, \bar{\mathcal{V}} \rangle] \tag{A.16}$$

and the following constraints hold

$$\langle \mathcal{V}, \partial_i \mathcal{V} \rangle = 0 \tag{A.17}$$

$$\langle \mathcal{V}, \bar{\mathcal{V}} \rangle = i \tag{A.18}$$

$$\langle U_i, \bar{U}_{\bar{j}} \rangle = i g_{i\bar{j}} \tag{A.19}$$

A.2 Special Kähler Quantities for $F = -2i\sqrt{X^0(X^1)^3}$

Given the prepotential, $F = -2i\sqrt{X^0(X^1)^3}$, we can compute the period matrix using (A.7) and (A.8). We considering only one vector multiplet ($n_V = 1$) and no hypermultiplets, with a real scalar z, defined as

$$z = \frac{X^1}{X^0} \tag{A.20}$$

We have that $\Lambda, \Sigma = 0, 1$. We concentrate on gaugings with Fayet-Iliopoulos (FI) terms, ξ_Λ, which determine the electric charges of the gravitini

$$e_\Lambda \equiv g_\Lambda \equiv g\xi_\Lambda \tag{A.21}$$

subject to the Dirac quantization in the presence of magnetic charges

$$g_\Lambda p^\Lambda = \pm 1 \tag{A.22}$$

where p^Λ are the magnetic charges. In this work we have indeed studied static and magnetically charged only black holes. Thus the theory is gauged only electrically and the FI terms can be thought of as the electric charges of the gravitino fields.
The components of the $F_{\Lambda\Sigma}$ matrix are given by

$$F_{00} = \frac{i}{2} z^{3/2}; \quad F_{01} = F_{10} = -\frac{3}{2} i\sqrt{z}; \quad F_{11} = \frac{3}{2} i \frac{1}{\sqrt{z}} \tag{A.23}$$

Therefore the period matrix is computed using (A.7)

$$\mathcal{N}_{\Lambda\Sigma} = i \begin{pmatrix} -\sqrt{\frac{(X^1)^3}{(X^0)^3}} & 0 \\ 0 & -3\sqrt{\frac{X^0}{X^1}} \end{pmatrix} = i \begin{pmatrix} -z^{3/2} & 0 \\ 0 & -\frac{3}{\sqrt{z}} \end{pmatrix} \tag{A.24}$$

We observe that we have indeed obtained a purely imaginary period matrix.

Given the sections X^Λ

$$X^\Lambda = \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} \tag{A.25}$$

we can derive the F_Λ sections using (A.5)

$$F_\Lambda = i \begin{pmatrix} -\sqrt{\frac{(X^1)^3}{X^0}} \\ -3\sqrt{X^0 X^1} \end{pmatrix} \tag{A.26}$$

Observation So far all the quantities we have calculated depended only on the ratio X^1/X^0 and not on the specific choice of the sections.

Requiring that $z = X^1/X^0$, we can rewrite the sections and their complex conjugate as

$$X^\Lambda = \begin{pmatrix} 1 \\ z \end{pmatrix} \qquad \bar{X}^\Lambda = \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} \tag{A.27}$$

and

$$F_\Lambda = \begin{pmatrix} -iz^{3/2} \\ -3i\sqrt{z} \end{pmatrix} \qquad \bar{F}_\Lambda = \begin{pmatrix} i\bar{z}^{3/2} \\ 3i\sqrt{\bar{z}} \end{pmatrix} \tag{A.28}$$

Therefore the Kähler potential [1] can be calculated using (A.16)

$$\begin{aligned} K(z,\bar{z}) &= -\log[i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda)] \\ &= -\log\left[i\left(1\left(-iz^{3/2}\right) + \bar{z}\left(-3i\sqrt{z}\right) - 1\left(i\bar{z}^{3/2}\right) - z\left(3i\sqrt{\bar{z}}\right)\right)\right] \\ &= -\log\left[z^{3/2} + 3z^{1/2}\bar{z} + 3z\bar{z}^{1/2} + \bar{z}^{3/2}\right] \\ &= -\log\left[\left(\sqrt{z} + \sqrt{\bar{z}}\right)^3\right] \end{aligned} \tag{A.29}$$

[1] The choice of the sections is not unique. It is only required that $X^1/X^0 = z$. In fact we could have leaved X^0 general and as a consequence we have obtained a potential of the form

$$K(z,\bar{z}) = -\log\left[\bar{X}^0 X^0 \left(\sqrt{z} + \sqrt{\bar{z}}\right)^3\right]$$

The resulting metric would have been the same as (A.31) since the metric $g_{i\bar{j}}$ is invariant under the so-called Kähler transformations

$$K \to K' = K + f(z) + \bar{f}(\bar{z})$$

If we are choosing real sections, then the Kähler potential becomes

$$K = -\log\left[\left(X^0\right)^2 8\sqrt{\left(\frac{X^1}{X^0}\right)^3}\right] = -\log\left[8\sqrt{\left(X^1\right)^3 X^0}\right]$$

Therefore

$$e^{-K} = 8\sqrt{\left(X^1\right)^3 X^0}$$

which can be written in general as

$$e^{-K} = \beta^2\sqrt{\left(X^1\right)^3 X^0}$$

$$K(z,\bar{z}) = -\log\left[\left(\sqrt{z}+\sqrt{\bar{z}}\right)^3\right] \tag{A.30}$$

Then the metric on the Kähler manifold is given by (A.4)

$$g_{i\bar{j}} = g_{z\bar{z}} = \frac{3}{4(\sqrt{z}+\sqrt{\bar{z}})\sqrt{z\bar{z}}} \tag{A.31}$$

Since we are considering a real scalar

$$z = \bar{z} \tag{A.32}$$

then

$$K(z,\bar{z}) = K(z) = -\log(8z^{3/2}) \tag{A.33}$$

and

$$g_{z\bar{z}} = g_{zz} = \frac{3}{16z^2} \tag{A.34}$$

The sections (A.2) computed using the proposed parametrization become

$$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} = e^{K/2}\begin{pmatrix} 1 \\ z \\ -i\sqrt{z^3} \\ -3i\sqrt{z} \end{pmatrix} \tag{A.35}$$

and the scalar potential can be calculated using

$$V_g = -3\left|\mathcal{L}\right|^2 + g^{i\bar{j}}\partial_i\mathcal{L}\partial_{\bar{j}}\bar{\mathcal{L}} \tag{A.36}$$

where

$$\mathcal{L} = \langle\mathcal{G},\mathcal{V}\rangle = g_\Lambda L^\Lambda = e^{K/2}g_\Lambda X^\Lambda = \bar{\mathcal{L}} \tag{A.37}$$

and

$$\partial_i\left(\mathcal{L}\right) = \partial_z\left(\mathcal{L}\right) = -\frac{-\sqrt{2}g_1 z + 3\sqrt{2}g_0}{16z^{7/4}} \tag{A.38}$$

and the symplectic vector

$$\mathcal{G} = (g^\Lambda, g_\Lambda)^T \tag{A.39}$$

parametrizes tha gauging, which is electric only in the case in analysis ($g^\Lambda = 0$).

Then

$$
\begin{aligned}
V_g &= -3\left|\mathcal{L}\right|^2 + g^{zz}\left(\partial_z \mathcal{L}\right)^2 \\
&= -3e^K\left(g_0^2\left(X^0\right)^2 + g_1^2\left(X^1\right)^2 + 2g_0g_1X^0X^1\right) + \frac{16z^2}{3}\left(\frac{-\sqrt{2}g_1z + 3\sqrt{2}g_0}{16z^{7/4}}\right)^2 \\
&= -\frac{1}{3}g_1^2\sqrt{z} - g_0g_1\frac{1}{\sqrt{z}} \\
&= -g^2\left(\frac{\xi_0\xi_1}{\sqrt{z}} + \frac{\xi_1^2}{3}\sqrt{z}\right)
\end{aligned} \tag{A.40}
$$

Note that there is no need to specify a specific form for X^Λ in order to calculate V_g as it depends only on the ratio.

$$
V_g = -g^2\left(\frac{\xi_0\xi_1}{\sqrt{z}} + \frac{\xi_1^2}{3}\sqrt{z}\right) \tag{A.41}
$$

It is straightforward to verify that V_g yields to a negative cosmological constant Λ. In fact

$$
\partial_z\left(V_g\right) = -g^2\left(\frac{-3\xi_0\xi_1 + \xi_1^2 z}{6z^{3/2}}\right) = 0 \quad \Longrightarrow \quad z_\infty = 3\frac{\xi_0}{\xi_1} \tag{A.42}
$$

Then

$$
V_g(z_\infty) = -3g^2\sqrt{\frac{4}{27}\xi_0\xi_1^3} = -\frac{3}{l_{\text{AdS}}^2} \tag{A.43}
$$

A useful quantity is given by the following matrix

$$
\mathcal{M} = \begin{pmatrix} \text{Im}\mathcal{N} + \text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1}\text{Re}\mathcal{N} & -\text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1} \\ -\text{Re}\mathcal{N}(\text{Im}\mathcal{N})^{-1} & (\text{Im}\mathcal{N})^{-1} \end{pmatrix} \tag{A.44}
$$

which in our case reduces to

$$
\mathcal{M} = \begin{pmatrix} \text{Im}\mathcal{N} & 0 \\ 0 & (\text{Im}\mathcal{N})^{-1} \end{pmatrix} = \begin{pmatrix} -z^{3/2} & 0 & 0 & 0 \\ 0 & -\frac{3}{\sqrt{z}} & 0 & 0 \\ 0 & 0 & -z^{-3/2} & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{z}}{3} \end{pmatrix} \tag{A.45}
$$

The matrix $\mathcal{M}$ is useful to compute V_{BH}, which is a quantity comparing in the Einstein equation of motion

$$
V_{BH} \equiv -\frac{1}{2}\mathcal{Q}^T\mathcal{M}\mathcal{Q} \tag{A.46}
$$

where $\mathcal{Q}$ is the symplectic vector containing the charges

$$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} \tag{A.47}$$

with p^Λ denoting the magnetic charges ad q_Λ the electric ones, which in the case at hand are set to zero.
Computing (A.46) gives

$$V_{BH} = -\frac{1}{2} p^\Lambda (\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}) p^\Sigma = \frac{1}{2}\left((p^0)^2 z^{3/2} + 3(p^1)^2 \frac{1}{\sqrt{z}} \right) \tag{A.48}$$

There is an equivalent definition tat uses the central charge defined as

$$\mathcal{Z} \equiv \langle \mathcal{Q}, \mathcal{V} \rangle \tag{A.49}$$

In this terms the black hole potential reads

$$V_{BH} = |D_i \mathcal{Z}|^2 + |\mathcal{Z}|^2 \tag{A.50}$$

For our case

$$\mathcal{Z} = \frac{i}{2\sqrt{2}} \left(p^0 z^{3/4} + 3p^1 z^{-1/4} \right) \tag{A.51}$$

$$\begin{aligned} D_i \mathcal{Z} &= \left(\partial_z + \frac{1}{2} \partial_z K \right) \mathcal{Z} \\ &= i e^{K/2} \left(\partial_z \left(p^0 z^{3/2} + 3p^1 z^{1/2} \right) + \partial_z K \left(p^0 z^{3/2} + 3p^1 z^{1/2} \right) \right) \\ &= \frac{3i}{8\sqrt{2}} \left(p^0 z^{-1/4} - p^1 z^{-5/4} \right) \end{aligned} \tag{A.52}$$

Then

$$\begin{aligned} V_{BH} &= |\mathcal{Z}|^2 + g^{zz} D_z \mathcal{Z} D_z \mathcal{Z} \\ &= \frac{1}{2} \left((p^0)^2 z^{3/2} + 3(p^1)^2 \frac{1}{\sqrt{z}} \right) \end{aligned} \tag{A.53}$$

in agreement with the result in (A.48).

A.3 Table of special Kähler quantities

	Quantity	Our convention
Symplectic sections	$\mathcal{V}$	$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} = e^{K/2} \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}$
Real sections	X^Λ	$X^\Lambda = \frac{1}{2\sqrt{2}} \left(a_\Lambda + \frac{b_\Lambda}{r} \right)$
Imaginary sections	$F_\Lambda = \partial_\Lambda F$	$F_\Lambda = -i \begin{pmatrix} \sqrt{\frac{(X^1)^3}{(X^0)^3}} \\ 3\sqrt{\frac{X^0}{X^1}} \end{pmatrix}$
Prepotential	F	$F = -2i\sqrt{X^0(X^1)^3}$
Scalar field (real)	z	$z = \frac{X^1}{X^0}$
Kähler potential	$K(z,\bar{z}) = K(z)$	$K(z) = -\log\left[8\sqrt{(X^1)^3 X^0}\right]$
Kähler metric	$g_{z\bar{z}} = g_{zz}$	$g_{zz} = \frac{3}{16z^2}$
Coefficients	a_Λ	$a_\Lambda = -2l_{AdS}\mathcal{G}^\Lambda$
	$\mathcal{G}^\Lambda$	$\mathcal{G}^\Lambda = \left(\mathrm{Im}\mathcal{N}^{-1}\right)^{\Lambda\Sigma}\Big\vert_{z_\infty} g_\Lambda$
Coefficients	b_Λ	*not constrained*
Period matrix	$\mathcal{N}$	$\mathcal{N} = \bar{F}_{\Lambda\Sigma} + 2i\frac{\mathrm{Im}(F_{\Lambda\Delta})X^\Delta \mathrm{Im}(F_{\Sigma\Phi})X^\Phi}{X^\Delta \mathrm{Im}(F_{\Delta\Phi})X^\Phi}$
	$F_{\Lambda\Sigma}$	$F_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma F$
Gaugings	$\mathcal{G}$	$\mathcal{G} = \begin{pmatrix} g^\Lambda \\ g_\Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ g\xi_\Lambda \end{pmatrix}$
Charges	$\mathcal{Q}$	$\mathcal{Q} = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} = \begin{pmatrix} p^\Lambda \\ 0 \end{pmatrix}$
Symplectic invariant	$\mathcal{L}$	$\mathcal{L} = \langle \mathcal{V}, \mathcal{G} \rangle = e^{K/2} g_\Lambda X^\Lambda = e^{K/2}\frac{1}{l_{AdS}}$
Symplectic invariant	$\mathcal{Z}$	$\mathcal{Z} = \langle \mathcal{Q}, \mathcal{V} \rangle$

B Analysis of BPS Conditions

B.1 BPS constraints

The one-quarter BPS solutions have to satisfy the following constraints

$$2U^2(r)h^2(r)\left(\mathrm{Im}(e^{-i\alpha}U^{-1}(r)\mathcal{V})\right)' = 8h^2(r)\mathrm{Re}(e^{-i\alpha}\mathcal{L})\mathrm{Re}(e^{-i\alpha}\mathcal{V}) - \mathcal{Q} - h^2(r)\Omega\mathcal{M}\mathcal{G} \tag{B.1}$$

$$(U(r)h(r))' = 2h(r)\mathrm{Im}(e^{-i\alpha}\mathcal{L}) \tag{B.2}$$

$$\alpha' + \mathcal{A}_r = -2U^{-1}(r)\mathrm{Re}(e^{-i\alpha}\mathcal{L}) \tag{B.3}$$

where α is the phase factor of the supersymmetry parameter and it is discussed later in this section, while $\mathcal{A}_r$ is the connection on the Kähler manifold defined as

$$\mathcal{A}_r \equiv \frac{i}{2}\left(\bar{z}^{\bar{j}'}\bar{\partial}_{\bar{j}}K - z^{i'}\partial_i K\right) \tag{B.4}$$

The last constraint is the so-called Dirac quantization constraint and states

$$\langle\mathcal{G},\mathcal{Q}\rangle = -\kappa \tag{B.5}$$

Identifying the phase factor

Following the prescription in [35], the phase factor α is identified by the following constraint

$$e^{2i\alpha} = \frac{\mathcal{Z} - ih^2\mathcal{L}}{\bar{\mathcal{Z}} + ih^2\bar{\mathcal{L}}} \tag{B.6}$$

where h^2 is the warp factor in the metric ansatz as in (2.6).
Evaluated upon the values of $\mathcal{Z}$ as in (A.51) and $\mathcal{L}$ as in (A.37), we obtain

$$e^{2i\alpha} = \frac{\frac{i}{2\sqrt{2}}\left(p^0 z^{3/4} + 3p^1 z^{-1/4}\right) - ie^{2(\Psi-U)}e^{K/2}g_\Lambda X^\Lambda}{-\frac{i}{2\sqrt{2}}\left(p^0 z^{3/4} + 3p^1 z^{-1/4}\right) + ie^{2(\Psi-U)}e^{K/2}g_\Lambda X^\Lambda} = -1 \tag{B.7}$$

The phase factor is then

$$\alpha = -\frac{\pi}{2} \tag{B.8}$$

B.1.1 Check of (B.1)

Let's verify (B.1). Using the definitions of $U(r)$ as in (2.7), $h(r)$ as in (2.8) and of $\mathcal{V}$ as in (A.2), we can write the LHS as:

$$2r^2 f \mathrm{Im}\left(i\frac{e^{-K/2}}{\sqrt{f}}e^{K/2}\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}\right)' \tag{B.9}$$

Since X^Λ are real, while F_Λ are purely imaginary, we get

$$LHS: \qquad 2r^2 f\left(\frac{1}{\sqrt{f}}\begin{pmatrix} X^0 \\ X^1 \\ 0 \\ 0 \end{pmatrix}\right)' \quad \text{with} \quad X^\Lambda = \frac{1}{2\sqrt{2}}\left(a_\Lambda + \frac{b_\Lambda}{r}\right) \tag{B.10}$$

Calculating the radial derivative we get for the LHS:

$$LHS: \qquad -\frac{1}{2\sqrt{2}\sqrt{f}}\left(\begin{pmatrix} rf'(a_0 r + b_0) + 2fb_0 \\ rf'(a_1 r + b_1) + 2fb_1 \\ 0 \\ 0 \end{pmatrix}\right) \tag{B.11}$$

Note that the first term on the RHS is automatically zero since from (B.33) $\mathcal{L}$ is real and $\alpha = -\frac{\pi}{2}$. Using the definitions for $\mathcal{Q}$ as in (A.47), Ω as in (A.15), $\mathcal{M}$ as in (A.45) and $\mathcal{G}$ as in (A.39), and remembering that we have magnetic charges only ($q_\Lambda = 0$)

and electric gaugings ($g^\Lambda = 0$), the RHS can be written as

$$\begin{aligned} RHS: \quad & -\begin{pmatrix} p^0 \\ p^1 \\ 0 \\ 0 \end{pmatrix} - e^{-K} r^2 \begin{pmatrix} 0 & 0 & z^{-3/2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{z}}{3} \\ -z^{3/2} & 0 & 0 & 0 \\ 0 & -\frac{3}{\sqrt{z}} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ g_0 \\ g_1 \end{pmatrix} \\ & = -\begin{pmatrix} p^0 \\ p^1 \\ 0 \\ 0 \end{pmatrix} - e^{-K} r^2 \begin{pmatrix} g_0 z^{-3/2} \\ \frac{\sqrt{z}}{3} g_1 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} p^0 + e^{-K} r^2 g_0 z^{-3/2} \\ p^1 + e^{-K} r^2 g_1 \frac{\sqrt{z}}{3} \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{B.12}$$

Remember that $z = X^1/X^0$.

So we should verify that the following two identities hold

$$-\frac{1}{2\sqrt{2}\sqrt{f}} \left(r f' a_0 (r + Q_0) + 2 a_0 f Q_0 \right) = -p^0 - e^{-K} r^2 g_0 \left(\frac{a_1 r + b_1}{a_0 r + b_0} \right)^{-3/2} \tag{B.13}$$

$$-\frac{1}{2\sqrt{2}\sqrt{f}} \left(r f' a_1 (r + Q_1) + 2 a_1 f Q_1 \right) = -p^1 - e^{-K} r^2 g_1 \frac{1}{3} \sqrt{\frac{a_1 r + b_1}{a_0 r + b_0}} \tag{B.14}$$

We can multiply both sides by $\sqrt{f}$ so not to have a square root at the denominator. We will use the following rewritings for a_0 and a_1 with respect to the definitions given in (2.22) and (2.23), respectively

$$a_\Lambda = -\sqrt{2} l_{AdS} \mathcal{G}^\Lambda \tag{B.15}$$

where

$$\mathcal{G}^\Lambda = \left(\mathrm{Im} \mathcal{N}^{-1} \right)^{\Lambda\Sigma} \Big|_{z_\infty} g_\Lambda \tag{B.16}$$

Upon evaluation we get

$$a_\Lambda = \left(\frac{1}{\sqrt{2} l_{AdS} g_0}; \frac{3}{\sqrt{2} l_{AdS} g_1} \right) \tag{B.17}$$

We report here the BPS values for the magnetic charges:

$$p^0 = -\frac{1}{4 g \xi_0} - \frac{2}{3} g b_1^2 \frac{\xi_1^2}{\xi_0}; \qquad p^1 = -\frac{3}{4} \frac{1}{g \xi_1} + \frac{2}{3} g \xi_1 b_1^2 \tag{B.18}$$

For convenience we rewrite them as

$$p^0 = -\frac{1}{g_0}\left(\frac{1}{4} + \frac{3Q_1^2}{l_{AdS}^2}\right), \qquad p^1 = -\frac{3}{4g_1}\left(1 - \frac{4Q_1^2}{l_{AdS}^2}\right) \tag{B.19}$$

We will use the extremal values for the constants c_1 and c_2 as, respectively, in (2.42) and (2.43).
We use the definition of f as in (2.14) rewritten as

$$f(r) = 1 + \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{1}{r^2 l_{AdS}^2}(r+Q_1)^3(r-3Q_1) \tag{B.20}$$

and its radial derivative is written as

$$f'(r) = -\frac{c_1}{r^2} - 2\frac{c_2}{r^3} + \frac{2}{l_{AdS}^2 r^3}\left(3Q_1^4 + 4Q_1^3 r + r^4\right) \tag{B.21}$$

Check of (B.13)

We should check that

$$\frac{1}{2\sqrt{2}}\frac{1}{\sqrt{2} l_{AdS} g_0}\left(rf'(r-3Q_1) - 6fQ_1\right) = \sqrt{f}\left(p^0 + e^{-K} r^2 g \xi_0 \left(\frac{a_1 r + b_1}{a_0 r + b_0}\right)^{-3/2}\right) \tag{B.22}$$

holds, where e^{-K} is as in (2.20) with $\beta = 1$. Using the above expressions for p^0 and a_Λ, it becomes

$$\frac{1}{4 l_{AdS} g_0}\left(rf'(r-3Q_1) - 6fQ_1\right) = \frac{\sqrt{f}}{g_0}\left(-\frac{1}{4} - \frac{3Q^2}{l_{AdS}^2} + \frac{(r-3Q_1)^2}{2l_{AdS}^2}\right)$$
$$\frac{\left(l_{AdS}^2 - 6Q_1^2 + 2r^2\right)\left(-l_{AdS}^2 + 6Q_1^2 - 12Q_1 r + 2r^2\right)}{8 l_{AdS}^3 r} = \sqrt{f}\left(\frac{-l_{AdS}^2 + 6Q_1^2 - 12Q_1 + 2r^2}{4l_{AdS}^2}\right) \tag{B.23}$$

In order not to have a nasty square root we evaluate the square of the expression:

$$LHS: \qquad \frac{1}{64 l_{AdS}^6 r^2}\left(l_{AdS}^2 - 6Q_1^2 + 2r^2\right)^2\left(-l_{AdS}^2 + 6Q_1^2 - 12Q_1 r + 2r^2\right)^2 \tag{B.24}$$

$$RHS: \qquad \frac{1}{64 l_{AdS}^6 r^2}\left(l_{AdS}^2 - 6Q_1^2 + 2r^2\right)^2\left(-l_{AdS}^2 + 6Q_1^2 - 12Q_1 r + 2r^2\right)^2 \tag{B.25}$$

Therefore the condition (B.13) holds.

Check of (B.14)

We should check that also

$$-\frac{1}{2\sqrt{2}\sqrt{f}}\left(rf'a_1(r+Q_1) + 2a_1 fQ_1\right) = -p^1 - e^{-K} r^2 g_1 \frac{1}{3}\sqrt{\frac{a_1 r + b_1}{a_0 r + b_0}} \tag{B.26}$$

holds. After inserting the values for p^1, K and a_Λ we get:

$$\begin{aligned}\frac{3}{4l_{AdS}g_1}\left(rf'(r+Q_1)+2fQ_1\right) &= \frac{\sqrt{f}}{g_1}\left(-\frac{3}{4}+\frac{3Q_1^2}{l_{AdS}^2}+\frac{3(r+Q_1)^2}{2l_{AdS}^2}\right)\\ \frac{\left(l_{AdS}^2-6Q_1^2+2r^2\right)\left(-l_{AdS}^2+6Q_1^2+4Q_1R+2r^2\right)}{8l_{AdS}^3 r} &= \sqrt{f}\frac{\left(-l_{AdS}^2+6Q_1^2+2r^2+4Q_1 r\right)}{4l_{AdS}^2}\end{aligned} \tag{B.27}$$

As before, we evaluate the square of the two sides, obtaining:

$$LHS: \qquad \frac{\left(l_{AdS}^2-6Q_1^2+2r^2\right)^2\left(-l_{AdS}^2+6Q_1^2+4Q_1R+2r^2\right)^2}{64l_{AdS}^6 r^2} \tag{B.28}$$

$$RHS: \qquad \frac{\left(l_{AdS}^2-6Q_1^2+2r^2\right)^2\left(-l_{AdS}^2+6Q_1^2+4Q_1R+2r^2\right)^2}{64l_{AdS}^6 r^2} \tag{B.29}$$

B.1.2 Check of (B.2)

Let's compute the LHS and RHS of (B.2)

$$LHS: \qquad \left(\sqrt{f}r\right)' = \frac{1}{\sqrt{f}}\left(f+\frac{r}{2}f'\right) \tag{B.30}$$

$$RHS: \qquad 2re^{-K/2}\mathcal{L} = 2re^{-K/2}g_\Lambda L^\Lambda \tag{B.31}$$

where $\mathcal{L}$ is as defined in (A.37) using the following sections

$$L^\Lambda = e^{K/2}\frac{1}{2\sqrt{2}}\left(a_\Lambda+\frac{b_\Lambda}{r}\right) \tag{B.32}$$

Then

$$\begin{aligned}\mathcal{L} &= e^{K/2}\frac{g}{2\sqrt{2}}\left(\xi_0\left(a_0+\frac{b_0}{r}\right)+\xi_1\left(a_1+\frac{b_1}{r}\right)\right) =\\ &= e^{K/2}\frac{g}{2\sqrt{2}r}\left(\xi_0 a_0\left(r+Q_0\right)+\xi_1 a_1\left(r+Q_1\right)\right) =\\ &= e^{K/2}\frac{g}{2\sqrt{2}r}\left(\frac{1}{\sqrt{2}l_{AdS}g}\left(r-3Q_1\right)+\frac{3}{\sqrt{2}l_{AdS}g}\left(r+Q_1\right)\right) =\\ &= e^{K/2}\frac{1}{4rl_{AdS}}\left(r-3Q_1+3r+3Q_1\right) =\\ &= e^{K/2}\frac{1}{l_{AdS}}\end{aligned} \tag{B.33}$$

We have used the rewriting (B.17). Then (B.2) becomes

$$\frac{1}{\sqrt{f}}\left(f+\frac{r}{2}f'\right) = 2re^{-K/2}e^{K/2}\frac{1}{l_{AdS}} \quad\Longrightarrow\quad \left(f+\frac{r}{2}f'\right) = \sqrt{f}\left(\frac{2r}{l_{AdS}}\right) \tag{B.34}$$

In order to get rid of the square root, we evaluate the square of the equivalence

$$\left(f + \frac{r}{2}f'\right)^2 = f\frac{4r^2}{l_{AdS}^2} \tag{B.35}$$

where f and f' are as in previous section and the extremal values for the constants c_1 and c_2 as, respectively, in (2.42) and (2.43), we find that the LHS and RHS are equal and precisely

$$LHS = RHS = \frac{\left(l_{AdS}^2 - 6Q_1^2 + 2r^2\right)^2}{l_{AdS}^4} \tag{B.36}$$

This concludes the check of (B.2) for the solutions from [2].

B.1.3 Check of (B.3)

This one is immediate

- the phase factor α is constant so $\alpha' = 0$
- the Kähler quantity $\mathcal{L}$ is real, so that Re $\left(e^{-i\alpha}\mathcal{L}\right)$ is zero
- since the scalar field is real the connection $\mathcal{A}_r$ is zero as well

C Einstein's Equations and Conventions

C.1 General Conventions

We are using the mostly minus signature $(+---)$ and we report here the definition of covariant derivative for covariant and controvariant vectors, respectively

$$\nabla_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda} A^\lambda \tag{C.1}$$

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda \tag{C.2}$$

where Γ is called the Christoffel symbol and is defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) \tag{C.3}$$

and the definiton of the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$

$$\left[\nabla_\mu, \nabla_\nu \right] A^\rho = R^\rho{}_{\sigma\mu\nu} A^\sigma \tag{C.4}$$

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{C.5}$$

C.2 Einstein's Equations

We rewrite here the action: the bulk and boundary parts

$$S = \int d^4x \sqrt{-g} \left(\frac{R}{2} + g_{zz} \partial_\mu z \partial^\mu z + \mathcal{I}_{\Lambda\Sigma} F^\Lambda_{\mu\nu} F^{\Sigma\mu\nu} - V_g \right) - \int d^3x \sqrt{h} \Theta \tag{C.6}$$

We are going to derive the Einstein's equations, because they are going to be useful to express all the quantities appearing in the action using the warp factors of the metric alone. We are using the metric ansatz given in (2.6)

Variation of the action with respect to $g_{\mu\nu}$ produces

$$\delta S = \int d^4x\sqrt{-g}\Big(\frac{1}{2}\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right) - \frac{1}{2}g_{\alpha\beta}g_{zz}\partial_\mu z\partial^\mu z - \frac{1}{2}g_{\alpha\beta}\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \\ + \frac{1}{2}V_g + g_{zz}\partial_\alpha z\partial_\beta z + 2\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\alpha\sigma}F^{\Sigma}{}_{\beta}{}^{\sigma}\Big)\delta g^{\alpha\beta} \tag{C.7}$$

Thus the Einstein's equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -g_{\mu\nu}V_g + g_{\mu\nu}g_{zz}\partial_\sigma z\partial^\sigma z - 2g_{zz}\partial_\mu z\partial_\nu z + g_{\mu\nu}\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\rho\sigma}F^{\Sigma\rho\sigma} - 4\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\rho}F^{\Sigma}{}_{\nu}{}^{\rho} \tag{C.8}$$

Given that the only non-zero component of $F^{\Lambda}_{\mu\nu}$ is $F^{\Lambda}_{\theta\phi}$ as in (2.10) and given the definition of the black hole potential V_{BH} as in (A.46), we have that [1]

$$\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} = -\frac{V_{BH}}{h^4} \tag{C.9}$$

We are following the derivation in [38] where the Riemann tensor is defined with an overall minus sign, so that

$$R \equiv -g^{\mu\nu}\left(\partial_\rho\Gamma^{\rho}_{\mu\nu} - \partial_\nu\Gamma^{\rho}_{\rho\mu} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\rho}\right) \tag{C.10}$$

For the metric (2.6) it is given by

$$R = \frac{2}{h^2}\left(1 - h^2U'^2 - U^2(h'^2 + 2hh'') - hU(4h'U' + hU'')\right) \tag{C.11}$$

where primed quantities are intended derivatives with respect to the radial coordinate.

In components the Einstein's equations are:

- tt - component:

$$\frac{\left(-1 + 2hh'UU' + U^2(h'^2 + 2hh'')\right)}{h^2} = -V_g + g_{zz}\partial_r z\partial^r z - \frac{V_{BH}}{h^4} \tag{C.12}$$

[1] The explicit derivation of the property is given by

$$\begin{aligned}\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} &= \mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma} = \\ &= \mathcal{I}_{\Lambda\Sigma}\left(F^{\Lambda}_{\theta\phi}F^{\Sigma}_{\rho\sigma}g^{\theta\rho}g^{\phi\sigma} + F^{\Lambda}_{\phi\theta}F^{\Sigma}_{\rho\sigma}g^{\phi\rho}g^{\sigma\theta}\right) = \\ &= 2\mathcal{I}_{\Lambda\Sigma}F^{\Lambda}_{\theta\phi}F^{\Sigma}_{\theta\phi}g^{\theta\theta}g^{\phi\phi} = \\ &= \frac{1}{2}p^{\Lambda}\mathcal{I}_{\Lambda\Sigma}p^{\Sigma}\sin^2\theta g^{\theta\theta}g^{\phi\phi} = -\frac{V_{BH}}{h^4}\end{aligned}$$

- rr - component

$$-\frac{-1+U^2h'^2+2hh'UU'}{h^2}=V_g+g_{zz}\partial_r z\partial^r z+\frac{V_{BH}}{h^4} \tag{C.13}$$

- $\theta\theta$ - component

$$-\frac{hU'^2+U^2h''+U(2h'U'+hU'')}{h}=V_g-g_{zz}\partial_r z\partial^r z-\frac{V_{BH}}{h^4} \tag{C.14}$$

- $\phi\phi$ - component

$$-\frac{hU'^2+U^2h''+U(2h'U'+hU'')}{h}=V_g-g_{zz}\partial_r z\partial^r z-\frac{V_{BH}}{h^4} \tag{C.15}$$

Adding the tt and the rr components we obtain

$$g_{zz}\partial_r z\partial_r z=-\frac{h''}{h} \tag{C.16}$$

Adding the rr with the $\theta\theta$ component we get

$$V_g=\frac{1}{2h^2}\left(1-\frac{1}{2}(U^2h^2)''\right) \tag{C.17}$$

Finally, adding together the $\theta\theta$ and the tt components we get

$$V_{BH}=\frac{1}{2}h^2\left(1-U^2h'^2-hh''U^2+h^2U'^2+h^2UU''\right) \tag{C.18}$$

We reproduced the results of [38], with the correct sign in (C.18) as already noted in [1].

D Extrinsic and Intrinsic curvature for the boundary metric

D.1 Intrinsic Curvature for the Boundary Metric

The boundary metric is denoted with $h_{\mu\nu}$ and it has the following form

$$ds^2_{\text{boundary}} = U^2(r)dt^2 - h^2(r)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \tag{D.1}$$

where $U^2(r)$ and $h^2(r)$ are as defined, respectively, in (2.7) and in (2.8).
The Ricci scalar, i.e. the curvature, is computed as follows

$$\mathcal{R}_{(3)} = h^{\mu\nu}\left(-\partial_\sigma \Gamma^\sigma_{\mu\nu} + \partial_\nu \Gamma^\sigma_{\mu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\right) \tag{D.2}$$

The only non zero Christoffel symbols for the $h_{\mu\nu}$ metric are the following

$$\Gamma^\theta_{\phi\phi} = -\sin(\theta)\cos(\theta); \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos(\theta)}{\sin(\theta)} \tag{D.3}$$

The evaluation of (D.2) gives the following result

$$\begin{aligned}
\mathcal{R}_{(3)} = {} & h^{tt}\left(-\partial_\sigma \Gamma^\sigma_{tt} + \partial_t \Gamma^\sigma_{t\sigma} - \Gamma^\sigma_{tt}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{t\sigma}\Gamma^\sigma_{t\rho}\right) + \\
& + h^{\theta\theta}\left(-\partial_\sigma \Gamma^\sigma_{\theta\theta} + \partial_\theta \Gamma^\sigma_{\theta\sigma} - \Gamma^\sigma_{\theta\theta}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{\theta\sigma}\Gamma^\sigma_{\theta\rho}\right) + \\
& + h^{\phi\phi}\left(-\partial_\sigma \Gamma^\sigma_{\phi\phi} + \partial_\phi \Gamma^\sigma_{\phi\sigma} - \Gamma^\sigma_{\phi\phi}\Gamma^\rho_{\sigma\rho} + \Gamma^\rho_{\phi\sigma}\Gamma^\sigma_{\phi\rho}\right) = \\
& = h^{\theta\theta}\left(\partial_\theta \Gamma^\phi_{\theta\phi} + \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\theta\phi}\right) + h^{\phi\phi}\left(-\partial_\theta \Gamma^\theta_{\phi\phi} - \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\theta\phi} + \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\theta\phi} + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi}\right) = \\
& = h^{\theta\theta}\left(\partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) + \frac{\cos\theta}{\sin\theta}\frac{\cos\theta}{\sin\theta}\right) + \\
& \quad + h^{\phi\phi}\left(\partial_\theta\left(\sin\theta\cos\theta\right) - \frac{\cos\theta}{\sin\theta}\left(\sin\theta\cos\theta\right)\right) = \\
& = -h^{\theta\theta} - h^{\phi\phi}\sin^2(\theta) = \\
& = \frac{2}{e^{-K}r^2} = 2h^{-2}(r)
\end{aligned} \tag{D.4}$$

$$\mathcal{R}_{(3)} = 2h^{-2}(r) \tag{D.5}$$

D.2 Extrinsic Curvature for the Boundary Metric

D.2.1 Equivalence of different definitions

Here we give all the different definitions of the extrinsic curvature tensor that can be found in various references and show their equivalence.
Beginning with the geometric definition (3.15) and using (3.2) we can write

$$\begin{aligned}\Theta_{\mu\nu} &= -h^{\alpha}_{\mu}\left(\delta^{\beta}_{\nu} + n^{\beta}n_{\nu}\right)\nabla_{\alpha}n_{\beta} = \\ &= -h^{\alpha}_{\mu}\nabla_{\alpha}n_{\nu} - h^{\alpha}_{\mu}n_{\nu}\underbrace{n^{\beta}\nabla_{\alpha}n_{\beta}}_{0} = \\ &= -h^{\alpha}_{\mu}\nabla_{\alpha}n_{\nu}\end{aligned} \tag{D.6}$$

where $n^{\beta}\nabla_{\alpha}n_{\beta} = 0$ as a consequence of

$$\begin{aligned}n^{\beta}\nabla_{\alpha}n_{\beta} &= \frac{1}{2}\nabla_{\alpha}\left(n^{\beta}n_{\beta}\right) = \\ &= \frac{1}{2}\nabla_{\alpha}\left(-1\right) = \\ &= 0\end{aligned} \tag{D.7}$$

Therefore

$$\Theta_{\mu\nu} = -h^{\alpha}_{\mu}\nabla_{\alpha}n_{\nu} \tag{D.8}$$

and we observe that it is a quantity defined on the hypersurface as the following property holds

$$n^{\mu}\Theta_{\mu\nu} = 0 \tag{D.9}$$

In fact

$$\begin{aligned}n^{\mu}\Theta_{\mu\nu} &= -n^{\mu}h^{\alpha}_{\mu}\nabla_{\alpha}n_{\nu} = \\ &= -n^{\mu}\left(\delta^{\alpha}_{\mu} + n^{\alpha}n_{\mu}\right)\nabla_{\alpha}n_{\nu} = \\ &= -n^{\alpha}\nabla_{\alpha}n_{\nu} - n^{\alpha}\underbrace{n^{\mu}n_{\mu}}_{-1}\nabla_{\alpha}n_{\nu} = \\ &= 0\end{aligned} \tag{D.10}$$

Moreover we can show that $\Theta_{\mu\nu}$ is a symmetric tensor using the fact that n^μ is hypersurface orthogonal.

$$\Theta_{\mu\nu} = \Theta_{\nu\mu} \tag{D.11}$$

In fact, since n_μ is hypersurface orthogonal it can been written as a gradient normal vector to the hypersurface defined as $f(x^\mu) = const$, where f is a smooth function.

$$n_\mu = N\partial_\mu f = N\nabla_\mu f \tag{D.12}$$

where N is the normalization factor. If we evaluate $n_\mu \nabla_\nu n_\rho$ we get

$$\begin{aligned} n_\mu \nabla_\nu n_\rho &= N\nabla_\mu f \nabla_\nu \left(N\nabla_\rho f\right) = \\ &= N\left(\nabla_\mu f\right)\left(\nabla_\nu N\right)\left(\nabla_\rho f\right) + N^2\left(\nabla_\mu f\right)\nabla_\nu\nabla_\rho f \end{aligned} \tag{D.13}$$

We notice that if we antisymmetrize the three indices μ, ν, ρ, then the above expression vanishes since the first term is symmetric in the μ and ρ indices, while the second term is symmetric in the ν and ρ indices thanks to the torsion free property of the Levi-Civita connection.
Thus the hypersurface orthogonality property leads us to

$$n_{[\mu}\nabla_\nu n_{\rho]} = 0 \tag{D.14}$$

Expanding the above and contracting with n^μ we directly end up with (D.11)

$$\begin{aligned} n^\mu\left(n_\mu\nabla_\nu n_\rho - n_\mu\nabla_\rho n_\nu + n_\nu\nabla_\rho n_\mu - n_\nu\nabla_\mu n_\rho + n_\rho\nabla_\mu n_\nu - n_\rho\nabla_\nu n_\mu\right) = 0 \\ -\nabla_\nu n_\rho + \nabla_\rho n_\nu + n_\nu \underbrace{n^\mu\nabla_\rho n_\mu}_{0} - n^\mu n_\nu\nabla_\mu n_\rho + n^\mu n_\rho\nabla_\mu n_\nu - n_\rho \underbrace{n^\mu\nabla_\nu n_\mu}_{0} = 0 \\ \cancel{\nabla_\nu n_\rho} + \cancel{\nabla_\rho n_\nu} - h^\mu_\nu\nabla_\mu n_\rho + \cancel{\nabla_\nu n_\rho} + h^\mu_\rho\nabla_\mu n_\nu + \cancel{-\nabla_\rho n_\nu} = 0 \end{aligned} \tag{D.15}$$

Then

$$\begin{aligned} h^\mu_\nu\nabla_\mu n_\rho &= h^\mu_\rho\nabla_\mu n_\nu \\ \Theta_{\nu\rho} &= \Theta_{\rho\nu} \end{aligned} \tag{D.16}$$

Therefore we can write

$$\begin{aligned} \Theta_{\mu\nu} &= -h^\alpha_\mu\nabla_\nu n_\alpha = \\ &= -h^\alpha_\mu\nabla_\alpha n_\nu = \\ &= -\left(\delta^\alpha_\mu + n^\alpha n_\mu\right)\nabla_\alpha n_\nu = \\ &= -\nabla_\mu n_\nu = \\ &= -\frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right) \end{aligned} \tag{D.17}$$

Furthermore assuming n^μ to be a geodesic, i.e. $n^\mu \nabla_\mu n^\nu = 0$ we can show

$$\begin{aligned}
\Theta_{\mu\nu} &= -\frac{1}{2}\left(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha\right)\left(\delta^\alpha_\mu \delta^\beta_\nu + \underbrace{\delta^\alpha_\mu n^\beta n_\nu}_{0} + \underbrace{n_\mu n^\alpha \delta^\beta_\nu}_{0} + \underbrace{n^\alpha n_\mu n^\beta n_\nu}_{0}\right) = \\
&= -\frac{1}{2}\left(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha\right)\left(\delta^\alpha_\mu + n^\alpha n_\mu\right)\left(\delta^\beta_\nu + n^\beta n_\nu\right) = \\
&= -\frac{1}{2}\left(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha\right) h^\alpha_\mu h^\beta_\nu = \\
&= -\frac{1}{2}\left(\underbrace{n^\rho \nabla_\rho g_{\alpha\beta}}_{0} + g_{\rho\nu}\nabla_\mu n^\rho + g_{\mu\rho}\nabla_\nu n^\rho\right) h^\alpha_\mu h^\beta_\nu = \\
&= -\frac{1}{2} h^\alpha_\mu h^\beta_\nu \mathcal{L}_n g_{\alpha\beta}
\end{aligned} \tag{D.18}$$

where we have used the fact that $g_{\mu\nu}$ is covariantly consant, $\nabla_\alpha g_{\mu\nu} = 0$ and in the first line we have added three terms that are zero when contacted with $\left(\nabla_\alpha n_\beta + \nabla_\beta n_\alpha\right)$ either because of $n^\mu \nabla_\mu n^\nu = 0$ or because of (D.7).
Finally we can show that

$$\begin{aligned}
\Theta_{\mu\nu} &= -\frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right) = \\
&= -\frac{1}{2}\left(g_{\rho\nu}\nabla_\mu n^\rho + g_{\mu\rho}\nabla_\nu n^\rho + \underbrace{\underbrace{n_\nu n^\rho \nabla_\rho n_\mu}_{0} + \underbrace{n_\mu n^\rho \nabla_\rho n_\nu}_{0}}_{n^\rho \nabla_\rho (n_\mu n_\nu)}\right) = \\
&= -\frac{1}{2}\left(h_{\rho\nu}\nabla_\mu n^\rho + \underbrace{n_\rho n_\nu \nabla_\mu n^\rho}_{0} + h_{\mu\rho}\nabla_\nu n^\rho + \underbrace{n_\rho n_\mu \nabla_\nu n^\rho}_{0} + n^\rho \nabla_\rho\left(h_{\mu\nu} - g_{\mu\nu}\right)\right) = \\
&= -\frac{1}{2}\left(h_{\rho\nu}\nabla_\mu n^\rho + h_{\mu\rho}\nabla_\nu n^\rho + n^\rho \nabla_\rho h_{\mu\nu}\right) = \\
&= -\frac{1}{2}\mathcal{L}_n h_{\mu\nu}
\end{aligned} \tag{D.19}$$

D.2.2 Trace of extrinsic curvature

In the calculations we are going to use the following definition

$$\Theta_{\mu\nu} = -\frac{1}{2}\left(\nabla_\mu n_\nu + \nabla_\nu n_\mu\right) \tag{D.20}$$

where n_μ is an outgoing normal vector to the boundary manifold $\partial\mathcal{M}$ taken as

$$n^\mu = \left(0, -\sqrt{-g^{rr}}, 0, 0\right)^T, \qquad n_\mu = \left(0, \sqrt{-g_{rr}}, 0, 0\right) \tag{D.21}$$

So that $n^{\mu}n_{\mu} = -1$ and n^{μ} is spacelike.
What we are interested in is the trace of (D.20), calculated as follows

$$
\begin{aligned}
\Theta &= \Theta_{\mu\nu}g^{\mu\nu} = \\
&= -\frac{1}{2}\left(\nabla_{\mu}n_{\nu} + \nabla_{\nu}n_{\mu}\right)g^{\mu\nu} = \\
&= -\frac{1}{2}\left(\partial_{\mu}n_{\nu} - \Gamma^{\lambda}_{\mu\nu}n_{\lambda} + \partial_{\nu}n_{\mu} - \Gamma^{\lambda}_{\mu\nu}n_{\lambda}\right)g^{\mu\nu} = \\
&= \Gamma^{\lambda}_{\mu\nu}n_{\lambda}g^{\mu\nu} - \partial_{r}n_{r}g^{rr} = \\
&= \Gamma^{r}_{tt}n_{r}g^{tt} + \Gamma^{r}_{rr}n_{r}g^{rr} + \Gamma^{r}_{\theta\theta}n_{r}g^{\theta\theta} + \Gamma^{r}_{\phi\phi}n_{r}g^{\phi\phi} - \partial_{r}n_{r}g^{rr}
\end{aligned}
\tag{D.22}
$$

The Christoffel symbols of ineterest are given by

$$
\begin{aligned}
\Gamma^{r}_{tt} &= \frac{1}{2}g^{r\alpha}\left(\partial_{t}g_{t\alpha} + \partial_{t}g_{\alpha t} - \partial_{\alpha}g_{tt}\right) = \\
&= -\frac{1}{2}g^{rr}\partial_{r}g_{tt} = e^{2K}f^{2}\left(\frac{K'}{2} + \frac{f'}{2f}\right)
\end{aligned}
\tag{D.23}
$$

$$
\begin{aligned}
\Gamma^{r}_{rr} &= \frac{1}{2}g^{r\alpha}\left(\partial_{r}g_{r\alpha} + \partial_{r}g_{\alpha r} - \partial_{\alpha}g_{rr}\right) = \\
&= \frac{1}{2}g^{rr}\partial_{r}g_{rr} = -\left(\frac{K'}{2} + \frac{f'}{2f}\right)
\end{aligned}
\tag{D.24}
$$

$$
\begin{aligned}
\Gamma^{r}_{\theta\theta} &= \frac{1}{2}g^{r\alpha}\left(\partial_{\theta}g_{\theta\alpha} + \partial_{\theta}g_{\alpha\theta} - \partial_{\alpha}g_{\theta\theta}\right) = \\
&= -\frac{1}{2}g^{rr}\partial_{r}g_{\theta\theta} = fr^{2}\left(\frac{K'}{2} - \frac{1}{r}\right)
\end{aligned}
\tag{D.25}
$$

$$
\begin{aligned}
\Gamma^{r}_{\phi\phi} &= \frac{1}{2}g^{r\alpha}\left(\partial_{\phi}g_{\phi\alpha} + \partial_{\phi}g_{\alpha\phi} - \partial_{\alpha}g_{\phi\phi}\right) = \\
&= -\frac{1}{2}g^{rr}\partial_{r}g_{\phi\phi} = fr^{2}\sin^{2}\theta\left(\frac{K'}{2} - \frac{1}{r}\right)
\end{aligned}
\tag{D.26}
$$

We have now all the ingredients to evaluate (D.22)

$$
\begin{aligned}
\Theta = e^{2K} f^2 \left(\frac{K'}{2} + \frac{f'}{2f}\right)\left(\frac{1}{e^{K/2}\sqrt{f}}\right)\left(\frac{1}{e^{K} f}\right) + \\
&+ \left(\frac{K'}{2} + \frac{f'}{2f}\right)\left(\frac{1}{e^{K/2}\sqrt{f}}\right)\left(-e^{K} f\right) + \\
&+ f r^2 \left(\frac{K'}{2} - \frac{1}{r}\right)\left(\frac{1}{e^{K/2}\sqrt{f}}\right)\left(-\frac{1}{e^{-K} r^2}\right) + \\
&+ f r^2 \sin^2\theta \left(\frac{K'}{2} - \frac{1}{r}\right)\left(\frac{1}{e^{K/2}\sqrt{f}}\right)\left(-\frac{1}{e^{-K} r^2 \sin^2\theta}\right) + \\
&- \partial_r \left(\frac{1}{e^{K/2}\sqrt{f}}\right)\left(-e^{K} f\right) = \\
= e^{K/2}\sqrt{f}\left(\frac{K'}{2} + \frac{f'}{2f}\right) + e^{K/2}\sqrt{f}\left(\frac{K'}{2} + \frac{f'}{2f}\right) + \\
&- e^{K/2}\sqrt{f}\left(\frac{K'}{2} - \frac{1}{r}\right) - e^{K/2}\sqrt{f}\left(\frac{K'}{2} - \frac{1}{r}\right) + e^{K/2}\sqrt{f}\left(\frac{K'}{2} + \frac{f'}{2f}\right) = \\
= e^{K/2}\sqrt{f}\left(\frac{f'}{2f} + \frac{2}{r} - \frac{K'}{2}\right)
\end{aligned}
\tag{D.27}
$$

$$
\Theta = e^{K/2}\sqrt{f}\left(\frac{f'}{2f} + \frac{2}{r} - \frac{K'}{2}\right) \tag{D.28}
$$

The tt component, which is needed in the computation of the regular part of the mass, is calculated as follows

$$
\begin{aligned}
\Theta_{tt} &= -\frac{1}{2}\left(\partial_t n_t - \Gamma^{\lambda}_{tt} n_\lambda + \partial_t n_t - \Gamma^{\lambda}_{tt}\right) = \\
&= \Gamma^{r}_{tt} n_r = \\
&= e^{2K} f^2 \left(\frac{K'}{2} + \frac{f'}{2f}\right)\left(\frac{1}{e^{K/2}\sqrt{f}}\right) = \\
&= \left(e^{K} f\right) e^{K/2}\sqrt{f}\left(\frac{K'}{2} + \frac{f'}{2f}\right) = \\
&= h_{tt} e^{K/2}\sqrt{f}\left(\frac{K'}{2} + \frac{f'}{2f}\right)
\end{aligned}
\tag{D.29}
$$

$$
\Theta^{tt} = h^{tt} e^{K/2}\sqrt{f}\left(\frac{K'}{2} + \frac{f'}{2f}\right) \tag{D.30}
$$

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