

Laplacian in Spherical Coordinates

We want to write the Laplacian functional

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (1)$$

in spherical coordinates

$$\begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases} \quad (2)$$

To do so we need to invert the previous transformation rules and repeatedly use the chain rule

$$\begin{aligned} \frac{\partial}{\partial x(r, \theta, \phi)} &= \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial y(r, \theta, \phi)} &= \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z(r, \theta, \phi)} &= \frac{\partial r}{\partial z} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z} \frac{\partial}{\partial \phi} \end{aligned} \quad (3)$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x^2(r, \theta, \phi)} &= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \right) = \\ &= \frac{\partial}{\partial x} \left(\frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x} \frac{\partial}{\partial \phi} \right) = \\ &= \left(\frac{\partial^2 r}{\partial x^2} \right) \frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial x} \frac{\partial r}{\partial x} \right) \frac{\partial^2}{\partial r^2} + \left(\frac{\partial r}{\partial x} \frac{\partial \theta}{\partial x} \right) \frac{\partial^2}{\partial r \partial \theta} + \left(\frac{\partial r}{\partial x} \frac{\partial \phi}{\partial x} \right) \frac{\partial^2}{\partial r \partial \phi} + \\ &\quad + \left(\frac{\partial^2 \theta}{\partial x^2} \right) \frac{\partial}{\partial \theta} + \left(\frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial x} \right) \frac{\partial^2}{\partial \theta^2} + \left(\frac{\partial \theta}{\partial x} \frac{\partial r}{\partial x} \right) \frac{\partial^2}{\partial \theta \partial r} + \left(\frac{\partial \theta}{\partial x} \frac{\partial \phi}{\partial x} \right) \frac{\partial^2}{\partial \theta \partial \phi} + \\ &\quad + \left(\frac{\partial^2 \phi}{\partial x^2} \right) \frac{\partial}{\partial \phi} + \left(\frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} \right) \frac{\partial^2}{\partial \phi^2} + \left(\frac{\partial \phi}{\partial x} \frac{\partial \theta}{\partial x} \right) \frac{\partial^2}{\partial \phi \partial \theta} + \left(\frac{\partial \phi}{\partial x} \frac{\partial r}{\partial x} \right) \frac{\partial^2}{\partial \phi \partial r} \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{\partial^2}{\partial y^2(r, \theta, \phi)} &= \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y} \right) = \\ &= \frac{\partial}{\partial y} \left(\frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial y} \frac{\partial}{\partial \phi} \right) = \\ &= \left(\frac{\partial^2 r}{\partial y^2} \right) \frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial y} \frac{\partial r}{\partial y} \right) \frac{\partial^2}{\partial r^2} + \left(\frac{\partial r}{\partial y} \frac{\partial \theta}{\partial y} \right) \frac{\partial^2}{\partial r \partial \theta} + \left(\frac{\partial r}{\partial y} \frac{\partial \phi}{\partial y} \right) \frac{\partial^2}{\partial r \partial \phi} + \\ &\quad + \left(\frac{\partial^2 \theta}{\partial y^2} \right) \frac{\partial}{\partial \theta} + \left(\frac{\partial \theta}{\partial y} \frac{\partial \theta}{\partial y} \right) \frac{\partial^2}{\partial \theta^2} + \left(\frac{\partial \theta}{\partial y} \frac{\partial r}{\partial y} \right) \frac{\partial^2}{\partial \theta \partial r} + \left(\frac{\partial \theta}{\partial y} \frac{\partial \phi}{\partial y} \right) \frac{\partial^2}{\partial \theta \partial \phi} + \\ &\quad + \left(\frac{\partial^2 \phi}{\partial y^2} \right) \frac{\partial}{\partial \phi} + \left(\frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial y} \right) \frac{\partial^2}{\partial \phi^2} + \left(\frac{\partial \phi}{\partial y} \frac{\partial \theta}{\partial y} \right) \frac{\partial^2}{\partial \phi \partial \theta} + \left(\frac{\partial \phi}{\partial y} \frac{\partial r}{\partial y} \right) \frac{\partial^2}{\partial \phi \partial r} \end{aligned} \quad (5)$$

$$\begin{aligned}
\frac{\partial^2}{\partial z^2(r, \theta, \phi)} &= \frac{\partial}{\partial z} \left(\frac{\partial}{\partial z} \right) = \\
&= \frac{\partial}{\partial z} \left(\frac{\partial r}{\partial z} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z} \frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z} \frac{\partial}{\partial \phi} \right) = \\
&= \left(\frac{\partial^2 r}{\partial z^2} \right) \frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial z} \frac{\partial r}{\partial z} \right) \frac{\partial^2}{\partial r^2} + \left(\frac{\partial r}{\partial z} \frac{\partial \theta}{\partial z} \right) \frac{\partial^2}{\partial r \partial \theta} + \left(\frac{\partial r}{\partial z} \frac{\partial \phi}{\partial z} \right) \frac{\partial^2}{\partial r \partial \phi} + \\
&\quad + \left(\frac{\partial^2 \theta}{\partial z^2} \right) \frac{\partial}{\partial \theta} + \left(\frac{\partial \theta}{\partial z} \frac{\partial \theta}{\partial z} \right) \frac{\partial^2}{\partial \theta^2} + \left(\frac{\partial \theta}{\partial z} \frac{\partial r}{\partial z} \right) \frac{\partial^2}{\partial \theta \partial r} + \left(\frac{\partial \theta}{\partial z} \frac{\partial \phi}{\partial z} \right) \frac{\partial^2}{\partial \theta \partial \phi} + \\
&\quad + \left(\frac{\partial^2 \phi}{\partial z^2} \right) \frac{\partial}{\partial \phi} + \left(\frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial z} \right) \frac{\partial^2}{\partial \phi^2} + \left(\frac{\partial \phi}{\partial z} \frac{\partial \theta}{\partial z} \right) \frac{\partial^2}{\partial \phi \partial \theta} + \left(\frac{\partial r}{\partial z} \frac{\partial \phi}{\partial z} \right) \frac{\partial^2}{\partial r \partial \phi}
\end{aligned} \tag{6}$$

where we have

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan \left(\sqrt{\frac{x^2 + y^2}{z^2}} \right) \\ \phi = \arctan \left(\frac{y}{x} \right) \end{cases} \tag{7}$$

Then we have

$$\frac{\partial}{\partial r} \rightarrow \left(\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} \right) \frac{\partial}{\partial r} = \frac{1}{r^2} \frac{\partial}{\partial r} \tag{8}$$

$$\frac{\partial^2}{\partial r^2} \rightarrow \left(\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 + \left(\frac{\partial r}{\partial z} \right)^2 \right) \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial r^2} \tag{9}$$

$$\frac{\partial}{\partial \theta} \rightarrow \left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} \right) \frac{\partial}{\partial \theta} = \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \tag{10}$$

$$\frac{\partial^2}{\partial \theta^2} \rightarrow \left(\left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial y} \right)^2 + \left(\frac{\partial \theta}{\partial z} \right)^2 \right) \frac{\partial^2}{\partial \theta^2} = \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \tag{11}$$

$$\frac{\partial}{\partial \phi} \rightarrow \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) \frac{\partial}{\partial \phi} = 0 \tag{12}$$

$$\frac{\partial^2}{\partial \phi^2} \rightarrow \left(\left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 + \left(\frac{\partial \phi}{\partial z} \right)^2 \right) \frac{\partial^2}{\partial \phi^2} = \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \tag{13}$$

The mixed derivative terms, $\frac{\partial^2}{\partial r \partial \theta}$, $\frac{\partial^2}{\partial r \partial \phi}$ and $\frac{\partial^2}{\partial \theta \partial \phi}$, cancel out.

Putting all the terms together we get

$$\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{r^2 \sin \theta} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} = \\
&= \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial r} r^2 \sin \theta \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right)
\end{aligned} \tag{14}$$

$$\nabla^2 = \frac{1}{r^2 \sin \theta} \left(\frac{\partial}{\partial r} r^2 \sin \theta \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \right) \tag{15}$$