# Multi-Probe Cosmology

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March 2023

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These notes are based on the course given at the Gran Sasso Science Institute (GSSI) by Riccardo Murgia in March 2023. The main objective is to present an overall picture of what are the state-of-the-art challenges in this field, especially for what concerns the cosmological tension in the measurements of the expansion rate of the Universe. Furthermore, the course focuses on giving a general overview of what are the main probes that allow us to investigate the Universe. The major reference on which all the content is based is the *Cosmology* book by Daniel Baumann.

# Contents

# 1 A Cosmologist's Starter Kit

## 1.1 FRW Metric and its properties

Spacetime metric can be written in the form:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1}$$

Assuming spatial homogenuity and isotropy the line element can be written as:

$$ds^2 = -c^2 dt^2 + a^2(t) dl^2, \tag{2}$$

where

$$dl^2 = \frac{dr^2}{1 - \frac{kr^2}{R_0^2}} + r^2 d\Omega^2 \quad \text{with} \quad k = \begin{cases} 0 & \text{flat} \\ -1 & \text{hyperbolic} \\ +1 & \text{spherical} \end{cases} . \tag{3}$$

Combining 2 and 3 we can write the full line element:

$$ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \frac{kr^2}{R_0^2}} + r^2 d\Omega^2 \right] . \tag{4}$$

We went from 10 independent components of $g_{\mu\nu}$ to a single function on time $a(t)$, **the scale factor**, and a constant, $R_0$, **the curvature scale today**.

- The metric satisfies the following rescaling symmetries:

$$\begin{aligned} a &\to \lambda a \\ r &\to r/\lambda \\ R_0 &\to R_0/\lambda \end{aligned} \tag{5}$$

  We can exploit this to set the value of the scale factor today at $t = t_0$ to be unity: $a(t_0) \equiv 1$.

- Redefining $r$ as:

$$d\chi = \frac{dr}{\sqrt{1 - \frac{kr^2}{R_0^2}}} \tag{6}$$

  we can write the line element in a very simple form:

$$ds^2 = -c^2 dt^2 + a^2(t) \left[ d\chi^2 + S_k^2(\chi) d\Omega^2 \right] \implies ds^2 = -c^2 dt^2 + a^2(t) dl^2 \tag{7}$$

  where

$$S_{\mathrm{k}}^2(\chi) \equiv \begin{cases} \dfrac{\chi}{R_0} & k = 0 \\ \sinh\left(\dfrac{\chi}{R_0}\right) & k = -1 \\ \sin\left(\dfrac{\chi}{R_0}\right) & k = +1 \end{cases} \tag{8}$$

- We can write the conformal time $\eta$:

$$d\eta = \frac{dt}{a(t)} \tag{9}$$

  and write the line elements as:

$$ds^2 = a^2(\eta)\left[-c^2 d\eta^2 + \left(d\chi^2 + S_k^2(\chi)d\Omega^2\right)\right] \tag{10}$$

The metric has been now factorized into a static part and a time-dependent conformal scale factor $a(\eta)$.

### 1.2 Hubble law

Starting from the relation between comoving and physical distance

$$r_{\text{physical}} = a(t)r_{\text{comoving}} \tag{11}$$

and taking the time derivative:

$$\begin{aligned}\frac{dr_{\text{physical}}}{dt} &= \frac{da}{dt}\vec{r}_{\text{comoving}} + a(t)\frac{d\vec{r}_{\text{comoving}}}{dt} \\ &= \frac{\dot{a}}{a}a\vec{r}_{\text{comoving}} + a(t)\frac{d\vec{r}_{\text{comoving}}}{dt} \\ &= H\vec{r}_{\text{physical}} + \vec{v}_{\text{peculiar}}\end{aligned} \tag{12}$$

we end up with the Hubble law:

$$\vec{v}_{\text{gal}} = H\vec{r}_{\text{physical}} + \vec{v}_{\text{peculiar}} \qquad \text{with} \qquad H \equiv \frac{\dot{a}}{a} \tag{13}$$

where $H$ is the **Hubble parameter**, $H\vec{r}_{\text{physical}}$ is known as the **Hubble flaw** (the velocity of the galaxy resulting from the expansion of space) and $\vec{v}_{\text{peculiar}}$ is the velocity measured by an observer who is comoving with the Hubble flaw.

### 1.3 Redshift and Distances

Given that due to the Universe's expansion any length scales as the scale factor, we have that energy scales as:

$$\lambda \propto a \implies E = \frac{h}{\lambda} \propto a^{-1} \tag{14}$$

This allows us to define the redshift. For $t_0 > t_1$:

$$\lambda_0 = \frac{a(t_0)}{a(t_1)}\lambda_1 \tag{15}$$

so that the wavelength we receive is stretched.
Now consider a galaxy at a distance $d$. Isotropy allows us to choose coordinates in which the light emitted by such a galaxy is traveling along the radial direction. Photons move along a null light-like geodesic, so for a light ray $ds^2 = 0$, which implies:

$$a^2(\eta)\left[-c^2 d\eta^2 + d\chi^2\right] = 0 \implies \pm c\Delta\eta = \Delta\chi(\eta) \tag{16}$$

where the minus sign is for incoming photons and the plus sign for the outgoing ones. Note the advantage of working with conformal time: in the $\chi - \eta$ coordinates light rays follow straight lines.
Given the following quantities:

1. emission time from a source $\eta_1$ (the signal duration interval is short and is $\delta\eta$)

2. arrival time at the telescope happens at $\eta_0 = \eta_1 + \delta\eta = \eta_1 + \frac{d}{c}$

3. the conformal duration of the signal is the same at the source and at the detector, the physical one instead is given by:

$$\begin{aligned} \delta t_1 &= a(\eta_1)\delta\eta \\ \delta t_0 &= a(\eta_0)\delta\eta \end{aligned} \tag{17}$$

We have that the light is emitted with wavelength $\lambda_1 = c\delta t_1$ and is observed with wavelength $\lambda_2 = c\delta t_2$:

$$\frac{\lambda_0}{\lambda_1} = \frac{c\delta t_1}{c\delta t_0} = \frac{a(\eta_0)}{a(\eta_1)} \implies 1 + z = \frac{1}{a(t_1)} \tag{18}$$

where $a(t_0) = 1$ by definition and where we have used the conventional definition of redshift as the fractional shift in wavelength:

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} \tag{19}$$

- If $z \ll 1$, which is true for nearby sources, we can expand around $t = t_0$:

$$a(t_1) = 1 + (t_1 - t_0)H_0 + \cdots \implies z = H_0(t_0 - t_1) + \frac{1}{2}(2 + q_0) H_0^2 (t_1 - t_0)^2 \tag{20}$$

  where

$$q_0 \equiv -\left(\frac{\ddot{a}}{aH^2}\right)\bigg|_{t=t_0} \tag{21}$$

  is the deceleration parameter. Therefore, at first order:

$$z = H_0 \underbrace{(t_0 - t)}_{d/c} + \cdots \implies v \equiv cz \simeq H_0 d \tag{22}$$

  where $v$ is the **recession velocity** and $v = H_0 d$ is known as the **Hubble-Lemaitre law**.

- If $z > 1$ we should carefully treat the notion of distance:

  Distances appearing in the metric are not observable, because they do not take into account that the Universe is expanding and that light takes a finite amount of time to reach us.

### 1.3.1 Luminosity distance (standard candles)

A way to measure distances is to use the so called standard candles. These are objects of known intrinsic brightness, so that their observed brightness can be used to determine their distances.
Consider a source at redshift $z$. The comoving distance to that object is defined as:

$$\chi(z) = c\int_{t_0}^{t_1} \frac{dt}{a(t)} = c\int_0^z \frac{dz'}{H(z')} \tag{23}$$

The observed isotropic flux emitted by this source, if the space was static, would be:

$$F = \frac{L}{4\pi\chi^2} \tag{24}$$

Since we need to take into consideration the expansion of the Universe:

1. $4\pi\chi^2 \to 4\pi a^2(t_0)d_M^2$ with $d_M$ the metric distance

2. the photon arrival rate is different from the photon emission rate: $\frac{a(t_1)}{a(t_0)} = \frac{1}{1+z}$

3. the emitted and observed energies are different by an additional $\frac{1}{1+z}$ factor

Therefore:

$$F = \frac{L}{4\pi d_M^2(1+z)^2} \equiv \frac{L}{4\pi d_L^2} \tag{25}$$

$$d_L(z) = (1+z)d_M(z) \tag{26}$$

#### 1.3.2 Angular diameter distance (standard rulers)

An alternative way to measure distances is using standard rulers, which are object of known physical size. The observed angular size of these objects depend then on the distance. The typical size of hot and cold spots of the CMB can be predicted theoretically and is therefore a standard ruler.
Suppose the transverse physical size of an object is $D$.In a static Euclidean space then we would expect its angular size to be:

$$\delta\theta = \frac{D}{\chi} \tag{27}$$

where we have assumed $\delta\theta \ll 1$ (in radians), which is a good assumption for cosmological objects in general. If we add expansion:

$$\delta\theta = \frac{D}{a(t_1)d_M} \equiv \frac{D}{d_A} \tag{28}$$

Notice that the observed angular size depends on the distance at the time $t_1$ when the light was emitted.

$$d_A = \frac{d_M}{1+z} \tag{29}$$

Pay attention that **luminosity distance and angular diameter distance are not independent**:

$$d_L(z) = (1+z)^2 d_A(z) \tag{30}$$

### 1.4 Dynamics of the FRW Universe: the Friedmann Equations

To study the evolution of the scale factor $a(t)$ we start from Einstein's equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{31}$$

where

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{0j} \\ T_{i0} & T_{ij} \end{pmatrix} = \begin{pmatrix} \textbf{scalar}\text{ (energy density)} & \textbf{vector}\text{ (momentum density)} \\ \textbf{vector}\text{(energy flux)} & \textbf{stress tensor} \end{pmatrix} \tag{32}$$

and making the following assumptions:

- homogenuity: $T_{00} = \rho(t)c^2$
- isotropy: $T_{0j} = T_{j0}$

- isotropy around a point $\vec{x}$: $T_{ij}(\vec{x}=0) \propto \delta_{ij} \propto g_{ij}(\vec{x}=0)$

we end up with the **perfect fluid** equation:

$$T^{\mu}_{\nu} = g^{\mu\lambda} T_{\lambda\nu} = \begin{pmatrix} -\rho c^2 & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix} \implies T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U_\mu U_\nu + P g_{\mu\nu} \tag{33}$$

with $U^\mu = (c, 0, 0, 0)$ for a comoving observer. Imposing the conservation of the stress-energy tensor $\nabla_\mu T^\mu_\nu = 0$ we end up with the continuity equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0 \tag{34}$$

This equation describes **energy conservation** in the cosmological context. The usual energy conservation in flat space as derived from Noether's theorem relies on the symmetry under time translation. This symmetry is broken for an expanding Universe. The usual notion of energy conservation does not hold and is replaced by 34.
Most cosmological fluids can be parametrized in terms of a constant equation of state

$$\omega \equiv \frac{P}{\rho c^2} \tag{35}$$

so that we can write:

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a} \implies \rho = a^{-3(1+\omega)} \tag{36}$$

We have:

- **matter** (non relativistic particles): $|P| \ll \rho c^2 \implies \rho \propto a^{-3}$
- **radiation** (relativistic particles): $\omega = 1/3 \implies \rho \propto a^{-4}$ (photons besides being diluted $V \propto a^{-3}$ get also redshifted $\propto a{-}1$)
- **dark energy**: $P = -\rho c^2 \implies \rho \propto a^0 = \text{const}$. It could be:
  - **vacuum energy** $T^{\text{vac}}_{\mu\nu} = -\rho_{\text{vac}} c^2 g_{\mu\nu}$, but $\rho_{\text{vac}}$ from QFT is $10^{60}\rho_\Lambda$ (the **cosmological constant problem**)
  - **cosmological constant**:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \implies T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda c^4}{8\pi G} g_{\mu\nu} \equiv -\rho_\Lambda c^2 g_{\mu\nu} \tag{37}$$

  - $\omega_{\text{DE}} \sim -1$: **varying equation of state**

## 1.5 Friedmann equations

From Einstein's equations we can derive the two Friedmann equations are:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right) \end{aligned} \tag{38}$$

In terms of $H$:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2} \tag{39}$$

We define the **critical energy density** as the one needed to have a flat universe:

$$\rho_{\text{crit},0} \equiv \frac{3H_0^2}{8\pi G} \tag{40}$$

where the 0 subscript denotes that the quantities are evaluated today at $t = t_0$.
In this way we can define the **abundance parameters** as:

$$\Omega_{i,0} = \frac{\rho_{i,0}}{\rho_{\text{crit},0}} \tag{41}$$

With this parametrization we can write the Friedmann's equations as:

$$H^2 = H_0^2 \left(\Omega_{\text{r}} a^{-4} + \Omega_{\text{M}} a^{-3} + \Omega_{\text{k}} a^{-2} + \Omega_\Lambda\right) \tag{42}$$

Evaluating both sides at $t_0$, with $a(t_0) = 1$, leads to:

$$\Omega_{\text{k}} + \Omega_{\text{M}} + \Omega_{\text{k}} + \Omega_\Lambda = 1 \tag{43}$$

One of the central goals in Cosmology is to measure the parameters appearing in 42 and hence determine the composition of the Universe.

**SUMMARY AND KEY POINTS**

A Universe that is both homogeneous and isotropic is described by the **Robertson-Walker metric**:

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1 - \frac{kr^2}{R_0^2}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

where $k = -1, 0, +1$ for, respectively, hyperbolic, flat and spherical space with radius of curvature today $R_0$.
Light from distant galaxies experiences the expansion of the Universe and as a consequence the wavelength is stretched according to the **redshift** value:

$$z \equiv \frac{\lambda_{\text{obs}} - \lambda_{\text{em}}}{\lambda_{\text{obs}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})} - 1 \implies z + 1 = \frac{1}{a}$$

The **velocity of galaxies**, in turn, undergoes both the effects of peculiar motion and the expansion of the Universe:

$$\vec{v}_{\text{gal}} = H\vec{r}_{\text{phys}} + \vec{v}_{\text{pec}} \quad \text{with} \quad H \equiv \frac{\dot{a}}{a}$$

The evolution of the **scale factor** $a$ is determined by the **Friedmann equations**:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2 R_0^2}$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

where $\rho$ represents the total energy density of the Universe. Requiring a flat Universe $k = 0$ we can write the **critical energy density**:

$$\rho_{\mathrm{crit}} \equiv \frac{3H_0^2}{8\pi G}$$

Each component of the Universe satisfies the **continuity equation**:

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$$

with $\omega = 0$ for baryons and dark matter (non-relativistic matter), $\omega = 1/3$ for radiation (relativistic matter) and $\omega = -1$ for dark energy. Since densities can be written as $\rho \propto a^{-3(1+\omega)}$, we can write the two Friedmann equations as one equation:

$$H^2 = H_0^2\left(\Omega_{\mathrm{r}} a^{-4} + \Omega_{\mathrm{M}} a^{-3} + \Omega_{\mathrm{k}} a^{-2} + \Omega_{\Lambda}\right)$$

with $H_0$ the **Hubble constant** and $\Omega_i \equiv \rho_i/\rho_{\mathrm{crit}}$ the dimensionless density parameters today.

# 2 Thermal History of the Universe

## 2.1 $H(z)$

Rewriting 42 in terms of redshift, we get:

$$H(z) = H_0\sqrt{\cancel{\Omega_\mathrm{r}(1+z)^4} + \Omega_\mathrm{M}(1+z)^3 + \cancel{\Omega_\mathrm{k}(1+z)^2} + \Omega_\Lambda} \tag{44}$$

where $\Omega_\mathrm{r}$ is negligible (BAO). Then we have from CMB observations $\Omega_\mathrm{r} \ll 1$, $\Omega_\mathrm{M} \sim 0.3$ (this includes both baryons ad cold dark matter) and $\Omega_\Lambda \sim 0.7$. Generally we can write:

$$H(z) \propto \frac{1}{\sqrt{\rho(z)}} \qquad \Longrightarrow \qquad H(z) = H_0\sqrt{\Omega_\mathrm{M}(1+z)^3 + \Omega_\Lambda} \tag{45}$$

On the other side, $\Omega_\Lambda$ starts dominating after $z = 2$. Studying $H(z)$ (see Eq. 45) allows us to determine whether the dark energy is really constant.

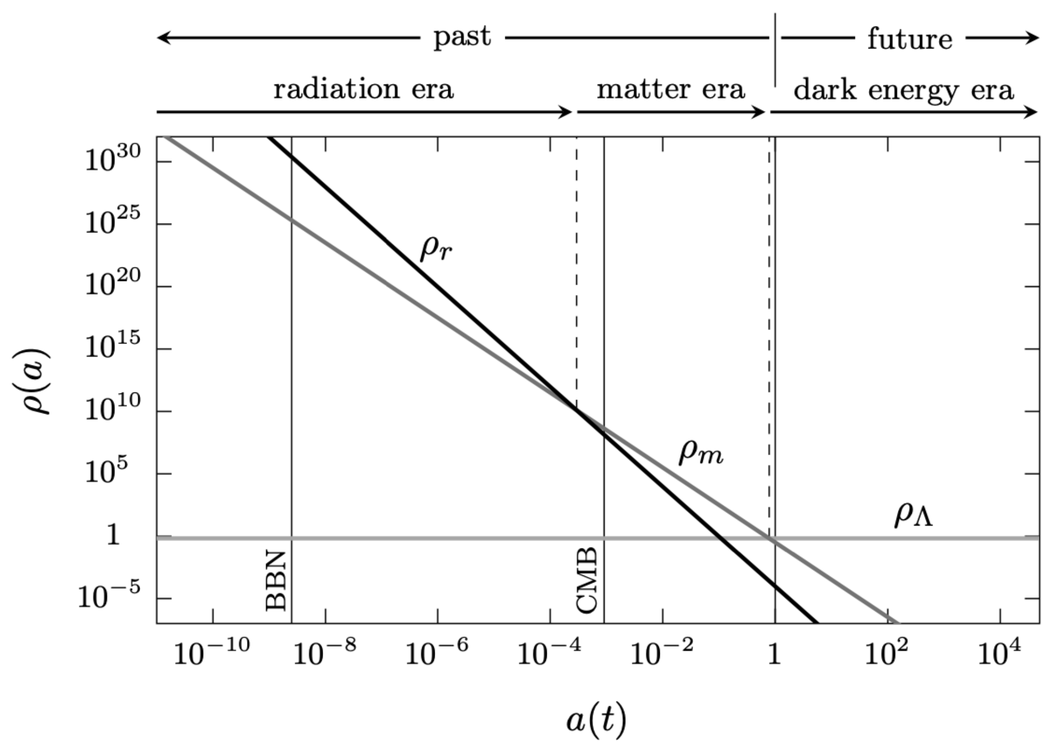
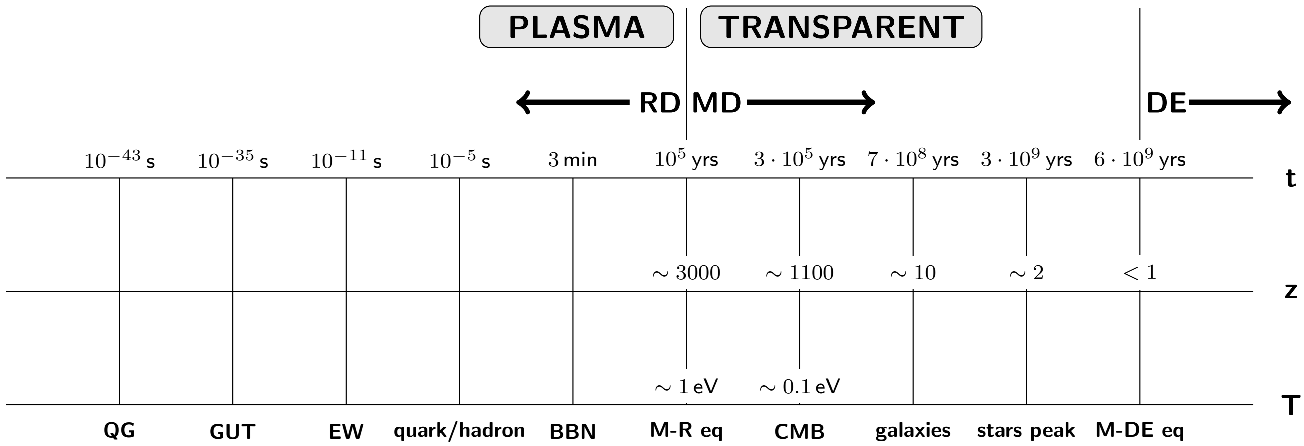



Figure 2.1: Evolution of the energy densities in the Universe. We can distinguish three epochs: radiation domination (RD), matter domination (MD) and dark energy domination (DED). Figure taken from *Cosmology* by Baumann (Fig. 2.10).

The interplay between different components allows us to write the **thermal history of the Universe**. The CMB spectrum is a strong observational evidence that the early Universe was in a state of **thermal equilibrium**.




## 2.2 Species evolution

The early Universe was a hot gas of weakly interacting particles. The description of such a system is better done using statistical mechanics.
We start from the probability distribution function of the momentum, $f(p,t)$, which can either follow Fermi-Dirac or Bose-Einstein statistics. The following quantities relate the microscopic properties to the macroscopic ones, respectively, the number density $n(T)$, the energy density $\rho(T)$ and the pressure $P(T)$:

$$\begin{aligned} n(T) &\propto \int d^3p f(p,t) \\ \rho(T) &\propto \int d^3p f(p,t) E(p) \\ P(T) &\propto \int d^3p f(p,t) \frac{p^2}{3E(p)} \end{aligned} \tag{46}$$

with $E(P) = \sqrt{p^2 + m^2}$. Each particle species has its own distribution function $f(p,t)$, number and energy densities and pressure. Species that are in thermal equilibrium share the same temperature $T$.
What we need to compare is the **interaction rate** $\Gamma$ with the **expansion rate** of the Universe $H$. At early times, during radiation domination era, we have that $\Gamma \gg H$: particles are in thermal equilibrium and relativistic (the temperature is much larger than the particle mass):

$$T \gg m \implies \rho \propto a^{-4}, \quad n \propto T^3 \implies \rho \propto T^4 \propto (1+z)^4 \tag{47}$$

In detail:

$$\begin{aligned} \rho &= \frac{\pi^2}{30} g_* T^4 \quad \text{energy density} \\ P &= \frac{\rho}{3} \quad \text{pressure} \\ S = \frac{\rho + P}{T} &= \frac{2\pi^2}{45} g_*^5 T^3 \quad \text{entropy} \end{aligned} \tag{48}$$

In the non relativistic limit (when the temperature drops below the particle mass), during MD era, we have:

$$T \ll m \implies n \propto \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} \tag{49}$$

$$\begin{aligned} \rho &\sim mn + \frac{3}{2}nT \\ P &= nT \end{aligned} \tag{50}$$

Moreover, when the Universe is radiation dominated, we can write the Friedmann's equation as $H^2 \propto T^4$, which gives us the relation between temperature and $H(z)$. As a rule of thumb we can write the following expression:

$$T \propto a^{-1}; \quad a \propto t^{1/2}; \quad \frac{T}{1\text{Mev}} \propto \left(\frac{1\text{s}}{t}\right)^{1/2} \tag{51}$$

### 2.3 Decoupling

We saw that when the temperature drops below a particle's mass, its number density exponentially decays (remember the $\exp(-m/T)$ factor). In order to survive till the present time, these particles must drop out of equilibrium much before the $m/T$ term becomes too large. This decoupling occurs when the interaction rate of particles becomes much smaller than the expansion rate of the Universe.
While $\Gamma \gg H$, the abundance of a certain species tracks its equilibrium value. Once $\Gamma$ becomes smaller than $H$, the species drops out of equilibrium and **decouples** from the plasma. To follow the evolution of each species we need to solve the **Boltzmann equation**:

$$\frac{d \log N_1}{d \log a} = -\frac{\Gamma}{H}[\ldots] \tag{52}$$

When a species decouples in the non relativistic regime, it does not interact any more and its abundance is frozen.

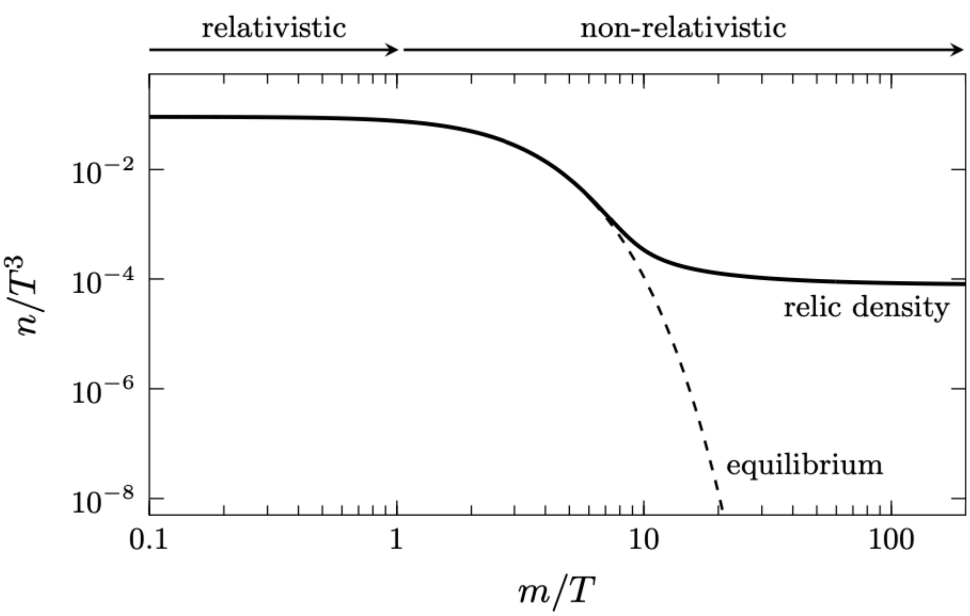



Figure 2.2: Scheme of how particle freeze-out happens. When the interaction rate drops below the expansion rate, the particles freeze out and maintain a constant relic density. Figure taken from *Cosmology* by Baumann (Fig. 3.9).

### 2.3.1 Dark matter freeze-out

Let us apply the Boltzmann equation to dark matter. This can also be a model of how dark matter was produced in the early Universe. Consider a heavy fermion $x$ that can annihilate with its anti-particle $\bar{x}$ to produce two light particles. The particles $x, \bar{x}$ may be dark matter, whereas $l$ could be particles of the Standard Model.

$$\underbrace{x + \bar{x}}_{\text{heavy particle/antiparticle (DM)}} \quad \leftrightarrow \quad \underbrace{l + \bar{l}}_{\text{light products (SM particles)}} \tag{53}$$

Then we make the following simplifying assumptions:

- The light particles $l$ are tightly coupled to the rest of the plasma so that they maintain their equilibrium densities
- We neglect any initial asymmetry between $x$ and $\bar{x}$ number densities
- We assume there is no other particle annihilation during the freeze-out of the $x$ species. This allows us to assume $T \propto a^{-1}$.

Given these assumptions the Boltzmann equation for the evolution of the $x$ particle can be written as:

$$\frac{1}{a^3}\frac{d(n_x a^3)}{dt} = \langle \sigma v \rangle \left[ n_x^2 - (n_x^{\text{eq}})^2 \right] \tag{54}$$

Changing variable to $Y_x \equiv \frac{n_x}{T^3} \propto N_x$, which is the number of particles in comoving volume, we end up with the **Riccati equation**:

$$\frac{dY_x}{dx} = -\frac{\lambda_i}{x^2} \left[ Y_x^2 - (Y_{\text{eq}})^2 \right] \tag{55}$$

where

$$\begin{aligned} x \equiv \frac{M_x}{T}; \qquad \frac{dx}{dt} = Hx \\ \lambda = \frac{\Gamma(M_x)}{H(M_x)} = \frac{M_x^3 \langle \sigma v \rangle}{H(M_x)} \end{aligned} \tag{56}$$

Note that $x$ is a convenient measure of time since the interesting moment is when $T \sim M_x$.
The Riccati equation can be solved numerically. Solving this equation for the decoupling of the weakly interacting particles, we obtain:

$$\Omega_x \sim \frac{10^{-8}\,\text{Gev}}{\langle \sigma v \rangle} \tag{57}$$

Observe that the observed dark matter density today depends inversely on the annihilation cross-section. This allows us to estimate the order of magnitude of the cross section, $\langle \sigma v \rangle \sim 10^{-8}\,\text{Gev}$, which leads to a temperature corresponding to the temperature of electro-weak transition. The fact that a thermal relic with a cross-section characteristic of the weak interaction gives the right order of magnitude for the observed dark matter abundance today is known as the **WIMP miracle**.

## SUMMARY AND KEY POINTS

We have sketched the essential points of the thermal history of the Universe for which the interplay between different components of the energy density has the major role. At early times, the **rate of particle interaction** $\Gamma$ is much larger than the **expansion rate** of the Universe $H$. Therefore, all the particles are in equilibrium at a common temperature $T$. The energy density is dominated by relativistic species.

To study non-equilibrium effects we use the **Boltzmann equation**. As long as the interaction rate is much larger than the expansion rate, the particle abundance tracks its equilibrium value. Once the interaction rate drops below the expansion rate, the particles drop out of thermal equilibrium and decouple from the thermal bath. Integrating the Boltzmann equation allows us to follow the non-equilibrium evolution of particle species: **dark matter freeze-out**, **BBN** and **recombination**

# 3 The Inhomogeneous Universe: LSS

To understand the large scale structure (LSS) of the Universe we need inhomogeneities and we need to track their evolution. When perturbations are small we can treat them as linear and use Newtonian gravity. Then we need to go in full GR.
For most of the history of the Universe, the growth of structure was dictated by the evolution of dark matter. Even though, we do not know yet the nature of dark matter, we only need two parameters, its equation of state and a sound speed, to capture its long-wavelength effects.

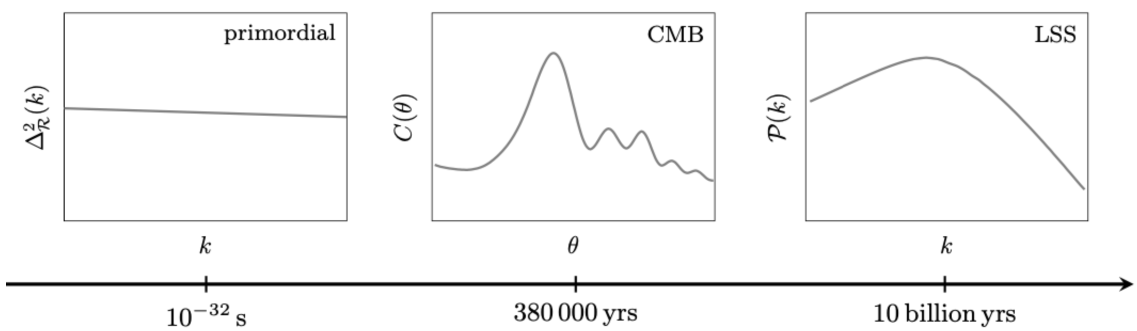



Figure 3.1: Evolution of the primordial curvature perturbations. Figure taken from *Cosmology* by Baumann (Fig. 6.2).

## 3.1 Structure growth in Newtonian theory: sub-horizon scales

We have a primordial spectrum of perturbations. How do we pass from this to what we actually observe? For all the sub-horizon scales, we can use Newtonian theory and do a numerical treatment, since we work with non relativistic fluids. We have the following equations:

**Pertubation Theory Equations**

1. **Continuity Equation**: $\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{u}) = 0$
2. **Euler Equation**: $\frac{d\vec{u}}{dt} + (\vec{u}\nabla)\vec{u} = -\frac{\nabla \rho}{\rho} - \nabla\phi$
3. **Poisson Equation**: $\nabla^2\phi = 4\pi G\rho$

These equations are meant for a static Universe. Adding expansion we can write the small perturbations around the background solution as:

$$\begin{aligned} \vec{r} &= a(t)\vec{x} \\ \rho &= \bar{\rho}(1+\delta) \\ \vec{u} &= Ha\vec{x} + \vec{v} \\ P &= \bar{P} + \delta P \\ \phi &= \bar{\phi} + \delta\phi \end{aligned} \tag{58}$$

As long as $\delta \ll 1$ we can linearize the fluid equations. Moreover, assuming that each perturbation evolves independently, we obtain the **linear perturbation equations**:

$$\begin{aligned} &1. \implies \delta = -\frac{1}{a}\nabla\vec{v} \\ &2. \implies \vec{v} + H\vec{v} = -\frac{1}{a\bar{\rho}}\nabla\delta P - \frac{1}{a}\nabla\delta\phi \\ &3. \implies \nabla^2\delta\phi = 4\pi G a^2\bar{\rho}\delta \end{aligned} \tag{59}$$

We can combine everything in one equation and obtain the time evolution of a linear density contrast:

- **In coordinate space**: $\ddot{\delta} + 2H\dot{\delta} - \left(\frac{c_s^2}{a^2}\nabla^2 + 4\pi G\bar{\rho}(t)\right)\delta = 0$
- **In Fourier space**: $\ddot{\delta}(k,t) + 2H\dot{\delta}(k,t) + c_s^2\left(\frac{k^2}{a^2} - k_J^2\right)\delta(k,t) = 0$

where $k_J$ is the wave number corresponding to the Jeans scale:

$$k_J \equiv \frac{\sqrt{4\pi G\bar{\rho}(t)}}{c_s(t)} \tag{60}$$

We have two extreme regions:

- At **small scales**: $\frac{k}{a} \gg k_J$. Here the pressure term dominates and we get the damped harmonic oscillator equation, the damping term being the expansion rate of the Universe $H$. We get oscillatory solutions with a decreasing amplitude.
- At **large scales**: $\frac{k}{a} \ll k_J$, the pressure term is negligible and we get the following equation:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0 \tag{61}$$

As we said, this treatment applies to non-relativistic fluids. Two important examples of these are cold dark matter and baryons after decoupling.

- For **baryons** the sound speed and the Jeans scale are strongly dependent on time. Before recombination, baryons are strongly coupled to photons in a single photon-baryon fluid. The Jeans scale is of order of the Hubble radius, so that there is no growth of baryonic fluctuations on sub-horizon scales. After recombination, the sound speed of baryons suddenly drops and its fluctuations start to grow.
- For **cold dark matter** the sound speed is negligible at all times, so that sub-horizon modes can start to grow earlier. The evolution of the growth depends on the evolution of the background density.

So we have that only fluctuations on scales larger than the Jeans scale can grow (due to the so called **Jeans instability**). Solving the above equations in the three eras leads us to the following results where we call **density contrast** the quantity $\delta \equiv \frac{\delta\rho}{\bar{\rho}}$:

- **RD**: $\delta \sim \ln(a)$, which means suppressed growth (**Meszaros effect**)
- **MD**: $\delta \sim a$, which means linear growth
- **DED**: $\delta \sim$ const, which means no growth at all.

### 3.2 Super-horizon scales

All of the results we just showed apply only to perturbations that are within the Hubble radius. To describe super-horizon perturbations we need to apply the full relativistic perturbation theory. We get that the super-horizon evolution of the density contrast is given by::

$$\delta \sim \begin{cases} a^2 & \text{RD} \\ a & \text{MD} \end{cases} \tag{62}$$

We can observe that:

- Modes entering the horizon during MD era grow with same behavior ($\propto a$) both for sub and super horizon
- In RD era, sub and super horizon modes behave very differently:

$$\begin{aligned} \text{sub-horizon} : \delta \propto \log a \\ \text{super-horizon} : \delta \propto a^2 \end{aligned} \tag{63}$$

  Since the moment of horizon entering depend on the wavenumber of the mode $k = aH$, we have a $k-$dependent growth of perturbations. This effect is described using the so called **transfer function** $T(k)$.

$$T(k) \equiv \frac{D_+(t_i)}{D_+(t)} \frac{\delta(k,t)}{\delta(k,t_i)} \tag{64}$$

  where $D_+$ is the scale-independent growth factor.

If denote by $k_{\text{eq}} \equiv (aH)_{\text{eq}}$ the wavenumber of the mode that entered the horizon at the time of matter-radiation equality, this scale separates modes that entered the horizon in the radiation era ($k > k_{\text{eq}}$) from those that entered in the matter domination era ($k < k_{\text{eq}}$). We can write the following transfer function:

$$T(k) \simeq \begin{cases} 1 & k < k_{\text{eq}} \\ \left(\frac{k_{\text{eq}}}{k}\right)^2 \ln\left(\frac{k_{\text{eq}}}{k}\right) & k > k_{\text{eq}} \end{cases} \tag{65}$$

### 3.3 Statistical properties

LSS in the Universe is not distributed randomly, but has interesting correlations between spatially separated points. By definition, the mean value of density perturbations is zero $\langle\delta\rangle = 0$. The first statistical non trivial measure of the density field is, at some fixed time $t$, the 2-point correlation function:

$$\xi(|x - x'|, t) \equiv \langle\delta(x,t)\delta(x',t)\rangle \tag{66}$$

In Fourier space:

$$\langle\delta(k,t)\delta^*(k',t)\rangle = (2\pi)^3\delta_D(k - k')P(k,t) \tag{67}$$

where $P(k,t)$ is the power spectrum. We have that $\xi(r)$ is easy to observe, whereas the power spectrum is easy to predict.

The linear evolution of Fourier modes we described earlier can be also written in terms of its power spectrum:

$$P(k,t) = T^2(k)\frac{D_+^2(t)}{D_+(t_i)}P(k,t_i) \tag{68}$$

The primordial power spectrum can also be written in the following form (**Harrison-Zel'dovich** spectrum):

$$P(k, t_i) = Ak^n \tag{69}$$

where $A$ and $n$ are constants. $n$ is called the spectral index. Harrison and Zel'dovich, before inflation was introduced, argued that the initial perturbations of our Universe were likely to have a power law with spectral index $n = 1$. This value is special as it means **scale invariance** (k-independence).
From observations from the CMB, instead, we have:

$$n = 0.9667 \pm 0.0040 \tag{70}$$

This fits perfectly to what is predicted from **inflation**. We expect, in fact, fluctuations that are close to scale invariance because the inflationary dynamics is approximately time-translation invariant. Since inflation has to end, we must have a small time-dependence in the evolution. This translates into a small deviation from a perfectly scale-invariant spectrum.

### 3.4 Matter power spectrum

Combining Eq. 65 and Eq. 69 we can predict the late-time **linear matter power spectrum**:

$$P(k, t) \propto \begin{cases} k^n & k < k_{\text{eq}} \\ k^{n-4} \ln\left(\frac{k}{k_{\text{eq}}}\right) & k > k_{\text{eq}} \end{cases} \tag{71}$$

Therefore, for a scale-invariant initial spectrum we expect the matter power spectrum to scale as $k$ on large scales and as $k^{-3}$ on small scales.
Taking into account relativistic perturbation theory, the non-linear clustering, at a certain time $\eta_0$ after the MR equality, reads as:

$$P(k, \eta_0) = \begin{cases} k^{-3} & k \ll k_0 \\ k & k_0 < k < k_{\text{eq}} \\ k^{-3} \ln\left(\frac{k}{k_{\text{eq}}}\right)^2 & k > k_{\text{eq}} \end{cases} \tag{72}$$

All sub-horizon modes grow as $a$ and the amount by which the modes will have grown depends on when they entered the horizon.

- For $k > k_{\text{eq}}$ modes that entered the horizon in the radiation era all evolve in the same way during the matter era, so the spectrum is uniformly boosted and no additional scale dependence is generated for these scales.
- Modes with $k < k_0$ are still outside the horizon and still today are unobservable
- On sub-horizon scales, for $k_0 < k < k_{\text{eq}}$, we get, as expected, the same results of the Newtonian approach

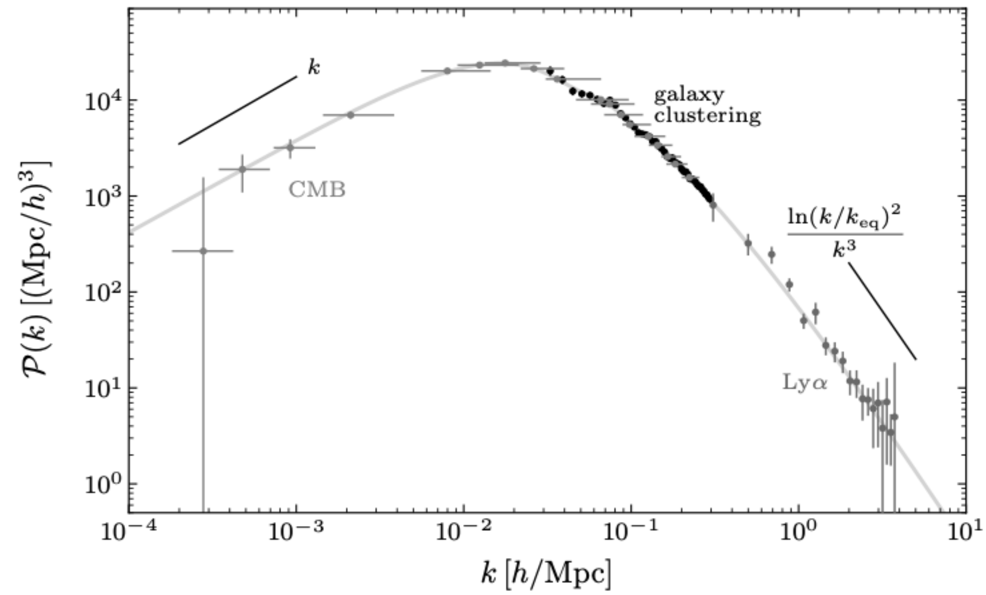



Figure 3.2: Measurements of the linear matter power spectrum. Figure taken from *Cosmology* by Baumann (Fig. 5.2).

**More about the 2-point galaxy correlation function**

If we take the correlation function of galaxies, it tells us the probability of finding a galaxy at a given distance. The 2-point galaxy correlation can be also written in physical space as:

$$\xi(r) = \frac{DD(r)\Delta r}{RR(r)\Delta r} - 1 \tag{73}$$

where $DD(r)\Delta r$ is the number of galaxy pairs within $r \pm \Delta r/2$ and $RR(r)\Delta r$ is what one would expect of galaxies were randomly distributed in space.
It assumes the following values:

$$\begin{cases} > 0 & \text{positive correlation} \\ = 0 & \text{uncorrelated} \\ < 0 & \text{anti correlation} \end{cases} \tag{74}$$

Typically we measure the correlation function in redshift space since it is easier observationally. Because of peculiar velocities $\xi(z) \neq \xi(r)$. this is known as redshift space distorsion (RSD).
We find that:

$$\xi(r) = \left(\frac{r}{r_0}\right)^{\gamma} \tag{75}$$

where $r_0 = 5.05\,\text{Mpc/h}$ and $\gamma \simeq 1.68$. Galaxies are:

- strongly clustered on scales $< 5\,\text{Mpc/h}$
- weakly clustered on scales $> 10\,\text{Mpc/h}$

## SUMMARY AND KEY POINTS

The growth of structure in **Newtonian theory** is valid for non-relativistic fluids (such as cold dark matter and baryons after decoupling) and on sub-horizon scales. For relativistic fluids and on super-horizon scales a full relativistic treatment is necessary.
In Newtonian theory we start from

1. **Continuity Equation**: $\frac{\partial\rho}{\partial t} + \nabla(\rho\vec{u}) = 0$
2. **Euler Equation**: $\frac{d\vec{u}}{dt} + (\vec{u}\nabla)\vec{u} = -\frac{\nabla\rho}{\rho} - \nabla\phi$
3. **Poisson Equation**: $\nabla^2\phi = 4\pi G\rho$

we incorporate the expansion of the Universe, then we write all the quantities as fluctuations around the homogeneous background solution and finally expand the above equations to linear order in these fluctuations.
The linearized equations can be combined into a single equation for the evolution of the density contrast:

$$\ddot{\delta} + 2H\dot{\delta} - \left(\frac{c_s^2}{a^2}\nabla^2 + 4\pi G\bar{\rho}(t)\right)\delta = 0$$

- On large scales pressure dominates and we get oscillatory solutions with a frequency set by the speed of sound.
- On small scales the pressure is negligible and gravity dominates. The resulting growth of instabilities is called the **Jeans instability**.

For dark matter instabilities, during RD, the expansion of space counteracts the gravitational instability and the growth of the perturbations is only logarithmic. During MD, instead, the perturbations start to grow linearly with the scale factor.
Observations measure the **spatial correlations** of the LSS. In linear theory, these correlations are easiest to predict in Fourier space, since the different Fourier modes evolve independently. The simplest correlation function in Fourier space is the power spectrum:

$$\langle\delta(k,t)\delta^*(k',t)\rangle = (2\pi)^3\delta_D(k-k')P(k,t)$$

For a Gaussian random field, the power spectrum contains all the statistical information of the density field. The late-time power spectrum can be written as:

$$P(k,t) = T^2(k)\frac{D_+^2(t)}{D_+(t_i)}P(k,t_i)$$

A nontrivial transfer function arises because of the suppressed growth of sub-horizon modes during the radiation era. The asymptotic scalings of the transfer function are:

$$T(k) \simeq \begin{cases} 1 & k < k_{\rm eq} \\ \left(\frac{k_{\rm eq}}{k}\right)^2 \ln\left(\frac{k_{\rm eq}}{k}\right) & k > k_{\rm eq} \end{cases}$$

The primordial power spectrum is:

$$P(k,t_i) = Ak^n$$

with $n \sim 1$ for the Harrison-Zel'dovich spectrum.
The matter power spectrum at late times, instead, is predicted using the full relativistic theory:

$$P(k, \eta_0) = \begin{cases} k^{-3} & k \ll k_0 \\ k & k_0 < k < k_{\text{eq}} \\ k^{-3} \ln\left(\dfrac{k}{k_{\text{eq}}}\right)^2 & k > k_{\text{eq}} \end{cases}$$

# 4 The Inhomogeneous Universe: CMB

## 4.1 The evolution of the photon-baryon fluid

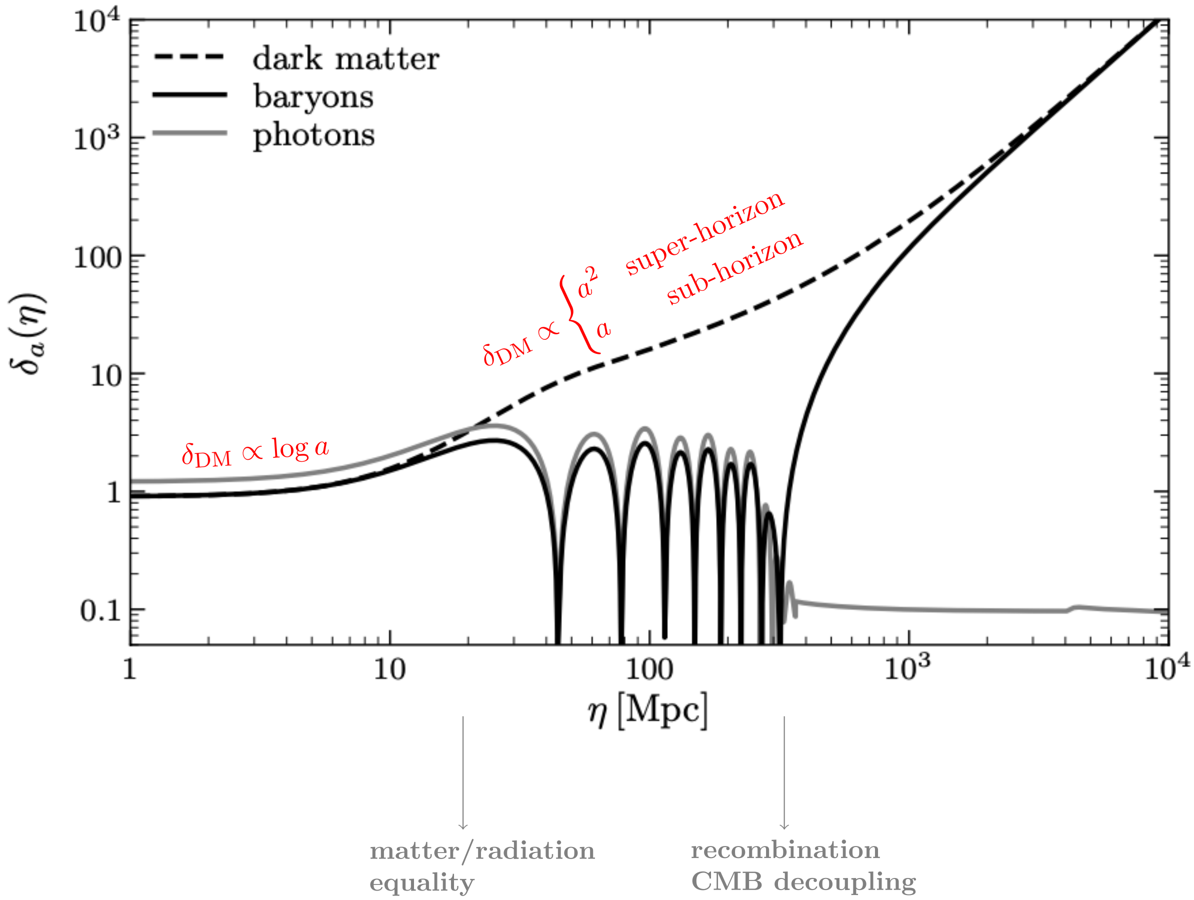



Figure 4.1: Evolution of densities contrasts for a representative Fourier mode. Initially the fluctuations of all components are of equal size and the baryons are tightly coupled to the photons. After decoupling the baryons lose the pressure support of the photons and fall into the gravitational potential wells created by dark matter. Note that around the time of decoupling the fluid approximation breaks down and the Boltzmann equations have to be solved numerically for the evolution of each species. Figure taken from *Cosmology* by Baumann (Fig. 6.8).

The analytic treatment we have seen so far, based on fluid approximation, breaks down around the time of CMB decoupling, $\eta_*$. The evolution equation of each species should be solved numerically (like with `CLASS` code). In the figure above Fig. 4.1 we have the summary of the evolution of overdensities as a function of conformal time. Note that it is crucial that the clustering of dark matter started before and then aided the growth of baryon perturbations. Without this assisted growth baryons would not have been able to cluster fast enough to explain the formation of galaxies.

Let us now study the evolution of photons and baryons before the decoupling. During matter-radiation equality, the photon-baryon fluid is tightly coupled (**tight coupling approximation**) and they are treated as a single photon-baryon fluid wuth velocity $\vec{v}_{\rm b} = \vec{v}_\gamma$. What we obtain is the equation of a forced harmonic oscillator:

$$\delta_\gamma^{''} + \underbrace{\frac{R'}{1+R}}_{\text{pressure}}\delta_\gamma^{'} - \frac{1}{3(1+R)}\nabla^2\delta_\gamma = \underbrace{\frac{4}{3}}_{\text{gravity}}\nabla^2\Psi + 4\phi^{''} + \frac{4R'}{1+R}\phi^{'} \tag{76}$$

where $R$ is the fractional contribution of baryons defined as:

$$R \equiv \frac{3}{4}\frac{\bar{\rho}_{\rm b}}{\bar{\rho}_\gamma} \propto \Omega_{\rm b}h^2 a(t) \implies \begin{cases} R \ll 1 & \text{RD} \\ R \simeq 1 & \text{MD} \end{cases} \tag{77}$$

This parameter is small at early times, but grows linearly with $a(t)$ and becomes of order $\sim 1$ around the time of recombination.
Equation 76 is the fundamental equation describing the evolution of perturbations in the photon-baryon fluid. From the pressure term we can read the **sound speed** of the photon-baryon fluid:

$$c_{\rm s}^2 \equiv \frac{1}{3(1+R)} \tag{78}$$

## 4.2 Cosmic sound waves

For simplicity, if we ignore gravity terms in Eq.76 and consider the solutions to the homogeneous equation:

$$\delta_\gamma^{''} + \frac{R'}{1+R}\delta_\gamma^{'} - \frac{1}{3(1+R)}\nabla^2\delta_\gamma = 0 \tag{79}$$

we can highlight two timescales: the expansion time and the oscillation time. On small scales, for $k \gg \frac{\mathcal{H}}{c}$ the oscillation time-scale is much smaller than the expansion rate. We can neglect gravity terms and adopt the WKB approximation to expand $R(t)$ and get the following solution:

$$\delta_\gamma(\eta, k) = C(k)\frac{\cos(kr_{\rm s})}{(1+r)^{1/4}} + D(k)\frac{\sin(kr_{\rm s})}{(1+R)^{1/4}} \tag{80}$$

where

$$r_{\rm s}(\eta) \equiv \int_0^\eta c_{\rm s}(\eta^{'})d\eta' \tag{81}$$

is the **sound horizon** at time $\eta$.

- During CMB, at the time of photon decoupling $\eta_*$:

$$r_{\rm s}(\eta_*) \simeq 145\,\text{Mpc} \tag{82}$$

- During BAO:

$$r_{\rm s}(\eta_{\rm drag}) \sim 147\,\text{Mpc} \tag{83}$$

The $C(k)$ and $D(k)$ functions are fixed by the super-horizon initial conditions.
At late times (high $k$ values) oscillations get damped (see Fig. 4.2) for 2 reasons:

1. **Diffusion (silk) damping**: photon diffusion from hot (high density) regions to cold (low density) regions erase temperature differences on small scales (the typical scales are at $k_{\rm silk} \sim 7\,\text{Mpc}$ and the corresponding multipole is $l_{\rm s} \sim 1300$)
2. **Landau damping**: recombination is not instantaneous; fluctuations that are smaller than the last scattering surface are somehow averaged, so that their amplitude is reduced

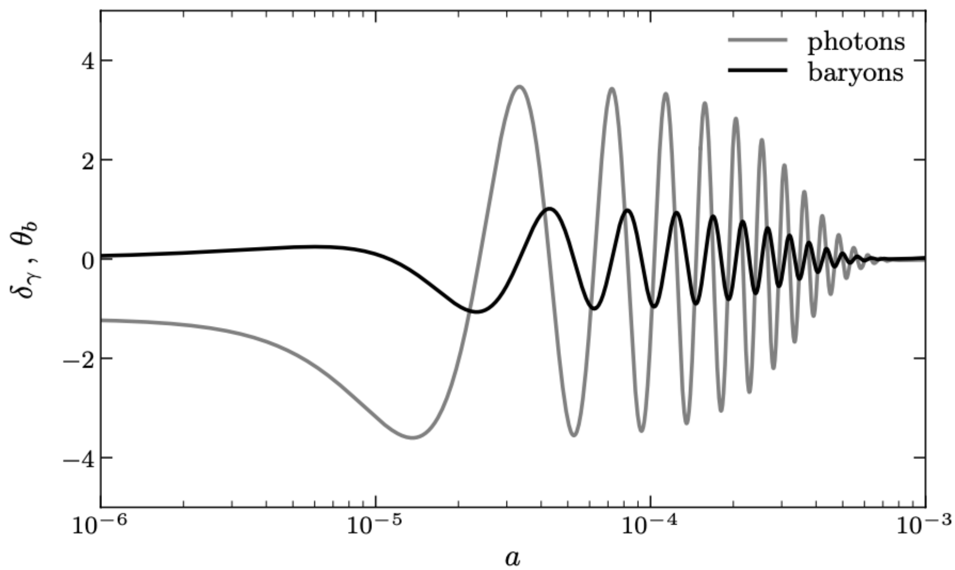



Figure 4.2: Evolution of density of photons (*gray*) and the velocity of baryons (*black*) for $k = 1\,h\text{Mpc}^{-1}$. Figure taken from *Cosmology* by Baumann (Fig. 6.9).

### 4.3 From primordial sound waves to CMB anisotropies

Fluctuations of different wavelength are captured (at decoupling): when we are looking the CMB we are seeing a snapshot of the primordial sound waves at the moment of photon decoupling. We see from Eq. 80 that the oscillation frequency of the photon-baryon system depends on the wavenumber $k$, meaning that different modes decouple at different moments in their evolution and therefore have different amplitudes at last scattering (see Fig. 4.3).

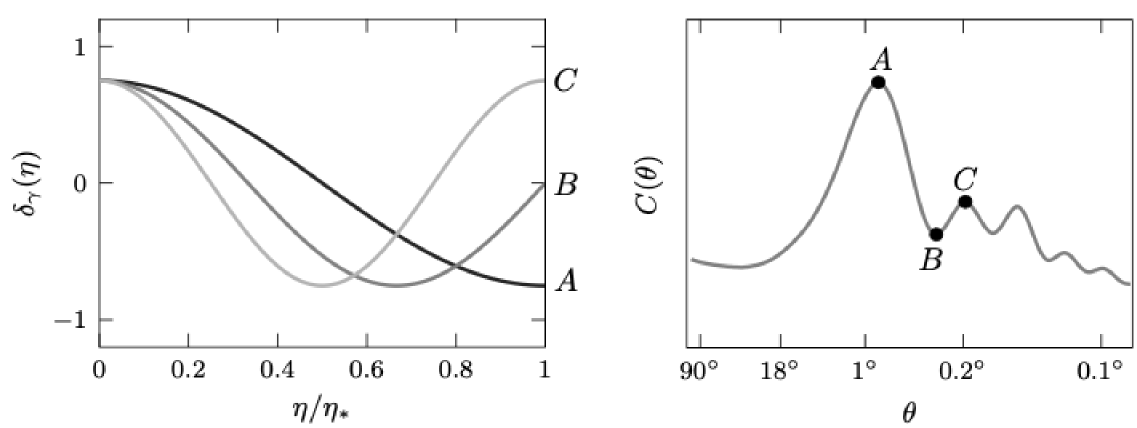



Figure 4.3: Illustration of the origin of peaks in the CMB power spectrum. On the *left* we see how different wavelengths are captured at different moments in their evolution and therefore have different amplitudes at decoupling. Since the square of the amplitude determines the power at a given length scale, waves that are captured at an extremum ($A$ or $C$) produce the peaks in the CMB spectrum, while waves that are captured with zero amplitude ($B$) produce the troughs. Figure taken from *Cosmology* by Baumann (Fig. 6.10).

Evaluating the solution at decoupling, $\eta = \eta_*$, gives the $k-$dependent amplitude of the fluctuations at last scattering. The $k$-dependent amplitude of $\delta_\gamma(\eta)$ at $\eta = \eta_*$ is fixed by:

$$k_\mathrm{n} = \frac{n\pi}{r_\mathrm{s}(\eta_*)} \tag{84}$$

which gives us the position of the peaks. These are the same peaks we observe in the matter power spectrum. However, there is no one-to-one correspondence between $k$ and $r$ spaces.
The power spectrum measures $|\delta_\gamma|^2$ so that waves *captured* in their extremes correspond to peaks in the spectrum (see $A$ ad $C$ in Fig. 4.3). Whereas waves with zero amplitude correspond to troughs (see $B$ in Fig. 4.3). We have that:

- The first peak corresponds to the **sound horizon**
- **Odd peaks** correspond to compressions in the plasma
- **Even peaks** correspond to rarefactions in the plasma

Note that the odd peak is always higher than the subsequent even one due to baryons falling in the compressed regions of the plasma (following the DM potential wells).
The positions of the peaks are determined by a combination of the sound horizon at decoupling and the angular distance to the surface of last scattering. The heights of the peaks carry information about the amount of dark matter and baryons in the Universe. The size of the damping on small scales depends on the density of neutrinos. This is how the measurements of the CMB anisotropies allow precise measurements of the key cosmological parameters.

### 4.3.1 Baryonic Acoustic Oscillations (BAO)

The same oscillations that are observed in the CMB anisotropies are imprinted in the distribution of baryonic matter. They leave an imprint in the clustering of galaxies. This happens because photons and baryons oscillate together before decoupling. Like photons, the different Fourier modes of the baryons decouple at different phases in their evolution. The initial conditions for the gravitational growth of the baryons include the oscillations of the photon-baryon fluid. This oscillatory feature gets transferred to the gravitational potential and the matter fluctuations, but the effect is smaller than in the CMB. Since correlations in the galaxy distribution are inherited from correlations in the matter density, these **baryonic acoustic oscillations** (BAO) are imprinted in the galaxy power spectrum.
The moment when the baryons are released from the drag of the photons is known as the **drag epoch**. Although photon and baryon decoupling are related they are not the same. Baryons decouple slightly later than photons at $z_\mathrm{drag} \sim 1020$ (compared to $z_* \sim 1090$). This is because there are much more photons than baryons in the Universe, so that photons stop feeling the effect of the baryons before the baryons stop experiencing the influence of the photons. After the drag epoch, baryons behave as pressureless matter.
We measure BAO both in galaxy clustering and in the L-$\alpha$ forest.

## 4.4 The CMB anisotropies

What makes the CMB such a powerful cosmological probe is the fact that the fluctuations were captured when they were still small and therefore they are accurately described by linear perturbation theory.
It is useful to separate the CMB spectrum (see Fig. 4.4) into three different regions:

- The **large-angle correlations** are sourced by fluctuations that were still super-horizon at recombination (therefore they did not evolve before CMB decoupling and are a direct probe of inflation).

- Perturbations with shorter wavelengths entered the horizon before recombination. Inside the horizon, the perturbations in the tightly coupled photon-baryon fluid propagate as **sound waves** supported by the large photon pressure. The oscillation frequency of these waves is a function of their wavelength and so different modes are captured at different moments in their evolution when the CMB was released at photon decoupling. This is the origin of the oscillatory pattern seen in the angular power spectrum.
- On scales smaller than the mean free pat of photons, the random diffusion of the photons can erase the density contrast in the plasma leading to a **damping** of the wave amplitudes. This suppresses the amplitude of the CMB power spectrum on small angular scales.

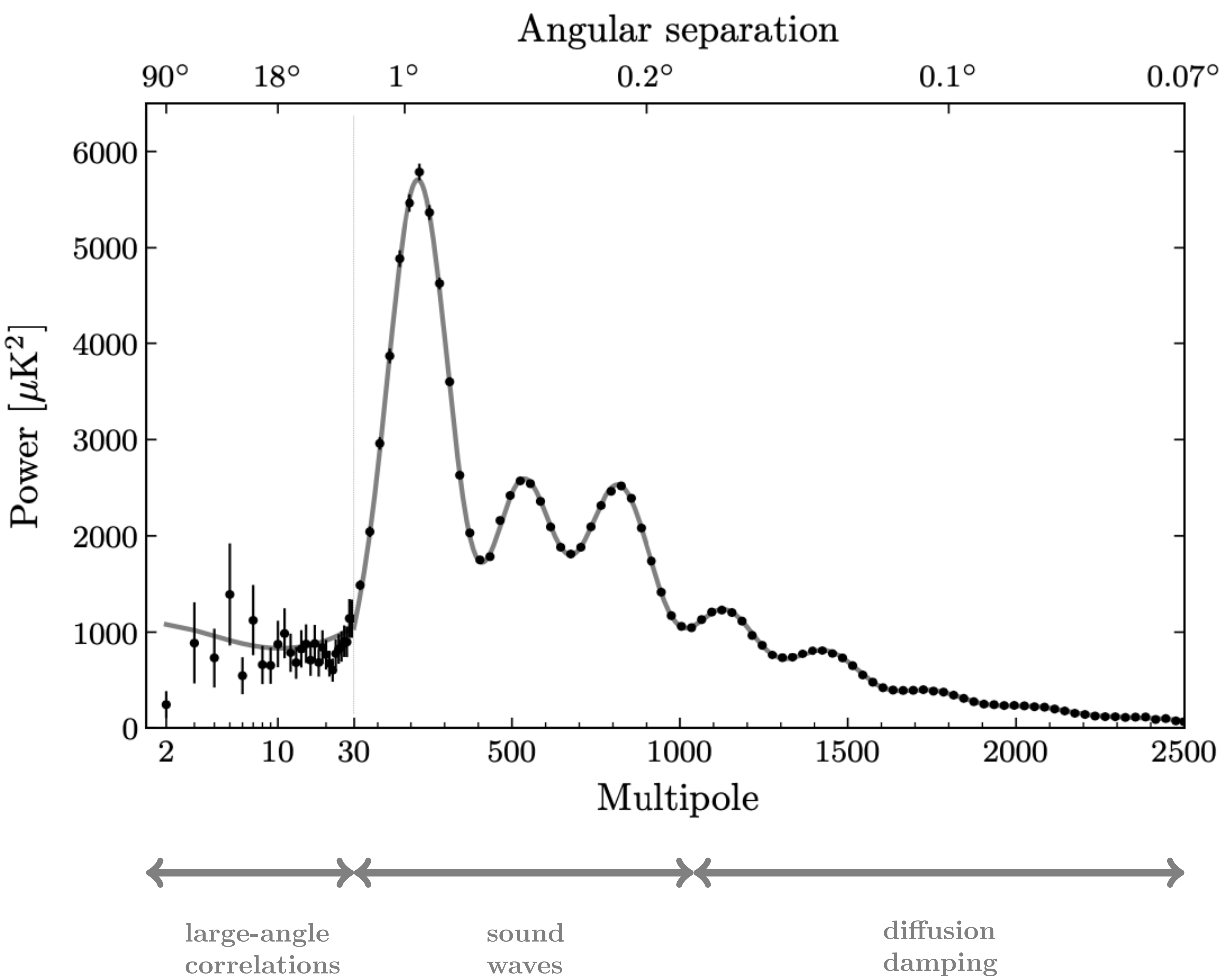
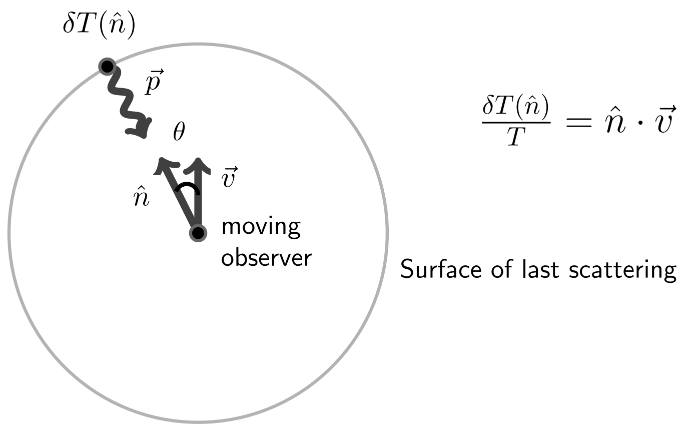



Figure 4.4: Planck measurements of the angular power spectrum of CMB temperature fluctuations. Figure taken from *Cosmology* by Baumann (Fig. 6.11).

### 4.5 CMB angular power spectrum

The largest anisotropy in the CMB is a temperature dipole due to the motion of the Solar System with respect to the rest frame of the CMB. Pay particular attention that the **underlying assumption** is that we remove the **dipole mode** due to kinematics, but we are not sure that it is all due to kinematics.




$$\frac{\delta T(\hat{n})}{T} = \hat{n} \cdot \vec{v} = v \cos\theta$$

Here $\hat{n}$ is the unit vector pointing at the observer's line of sight (which is opposite to the momentum $\vec{p}$ of the incoming radiation). $\vec{v}$ is the observer's velocity, i.e. the orbital velocity of the solar system plus the galaxy velocity with respect to the center of mass of the local group plus the velocity of the local group with respect to the CMB rest frame.
The problem is the discrepancy between the CMB estimation of the kinematic dipole and that from quasars ($\sim 5\sigma$ tension) and from SnIa ($\sim 2-3\sigma$ tension).

We can start from the primordial power spectrum (the same one of the matter distribution) and then deduce the one at later times. What we obtain is:

$$T(\hat{n}) \equiv \bar{T}_0(1+\theta(\hat{n}) \quad \text{with} \quad \theta(\hat{n}) = \frac{\delta T(\hat{n})}{\bar{T}_0} \tag{85}$$

In this case the 2-point correlation function depends on the angle $\theta$:

$$\theta(\hat{n}) = \sum_{l=2} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{n}) \tag{86}$$

with $a_{lm}$ the multiple moments and $Y_{lm}$ the spherical harmonics. Averaging over the whole Universe:

$$C(\theta) = \langle \theta(\hat{n}) \theta^*(\hat{n}') \rangle \tag{87}$$

Exploiting spherical harmonics properties we end up with:

$$C_l = 2\pi \int_{-1}^{+1} d\cos(\theta) C(\theta) P_l(\cos\theta) \tag{88}$$

with $P_l$ being the Legendre polinomiums.We also find a nice relation between $T_0$ and $C_l$:

$$\Delta_{\mathrm{T}}^2 \equiv \frac{l(l+1)}{2\pi} C_l T_0^2 \tag{89}$$

Typically this is what is plotted as the CMB spectrum. For a fixed $l$, we have $(2l+1)$ different $a_{lm}$s, i.e. $(2l+1)$ independent estimates of the true $C_l$s. Assuming we have extracted all the multipole moments $a_{lm}$s we can estimate the true $C_l$s as:

$$\hat{C}_l \equiv \frac{1}{2l+1} \sum_m |a_{lm}|^2 \quad \text{and} \quad \frac{\Delta C_l}{C_l} = \frac{\sqrt{\langle C_l - C_l^2 \rangle}}{C_l} = \sqrt{\frac{2}{2l+l}} \tag{90}$$

where $\frac{\Delta C_l}{C_l}$ is the intrinsic error on the $C_l$ due to the limited data sample and it is called **cosmic variance**.

What happens to $P(k, t_0)$? It evolves and the evolution tells us all the properties of the photon-baryon fluid:

$$\begin{pmatrix} \delta_\gamma \\ \Psi \\ v_\mathrm{b} \end{pmatrix}_{\eta_*, REC} \implies \begin{pmatrix} \text{free streaming} \\ \text{projection effects} \end{pmatrix} \implies \delta T(\hat{n})_\mathrm{today} \tag{91}$$

Let us now write the evolution equation:

$$\frac{\delta T}{T}(\hat{n}) = \underbrace{\left(\frac{\delta_\gamma}{4} + \Psi\right)_{\eta_*}}_{\text{Sachs-Wolf}} - \underbrace{(\hat{n} v_\mathrm{b})_{\eta_*}}_{\text{Doppler}} + \underbrace{\int_{\eta_*}^{\eta_0} d\eta (\phi' + \Psi')}_{\text{integrated Sachs-Wolf}} \tag{92}$$

- **Sachs-Wolf effect**: happens at the surface of last scattering. The $\delta_\gamma/4$ term tells us the intrinsic photon density fluctuation. The $\Psi$ term is related to the additional temperature perturbation due to gravitational redshift of photons (climbing out of the potential well inside the last scattering surface). The cause are the inhomogeneites in the density field at recombination.
- **Doppler effect**: due to the continuous scattering between free electrons and photons on the last scattering surface. This shift leads to an additional temperature fluctuation.
- **Integrated Sachs-Wolf effect**: this is an additional gravitational redshift due to the fact that gravitational wells are evolving with time in the DE-dominated era, as well as when radiation was still non negligible.

During RD the early integrated Sachs-Wolf effect increases the height of the first peak of the spectrum. During DED, the late integrated Sachs-Wolf effect, instead, has an impact on the large scales leading to more power at low $l$s. The Doppler effect reduces the contrast between the valleys and the throats.

## 4.6 Impact of cosmological parameters on the CMB

In the ΛCDM model we have 6 parameters:

$$\{A_\mathrm{s}, n_\mathrm{s}, \omega_\mathrm{b}, \omega_\mathrm{M}, \Omega_\Lambda, \tau\} \tag{93}$$

where the first two parameters represent the amplitude and the slope of the primordial power spectrum and $\tau$ is the optical depth at reionization (a late Universe parameter that can be substituted with $z_\mathrm{REI}$). Remember then that the ΛCDM model is flat by definition.
Instead of using $\Omega_\Lambda$ as a free parameter, we can use $h$:

$$h = \sqrt{\frac{\omega_\mathrm{M}}{1 - \Omega_\Lambda}} \implies h = \frac{H_0}{100 \frac{\mathrm{km}}{sMpc}} \sim 0.7 \tag{94}$$

Or better we could use the **angular size of the sound horizon** at recombination $\theta_\mathrm{s}$:

$$\theta_\mathrm{s} = \frac{r_\mathrm{s}(z_\mathrm{CMB})}{d_\mathrm{A}(z_\mathrm{CMB})} = \frac{\int_\infty^{z_\mathrm{CMB}} dz \frac{c_\mathrm{s}(z)}{\sqrt{\rho_\mathrm{TOT}(z)}}}{\int_0^{z_\mathrm{CMB}} \frac{dz}{\sqrt{\rho_\mathrm{TOT}(z)}}} \tag{95}$$

where the numerator regards pre-recombination physics $(\omega_\mathrm{b}, \omega_\mathrm{M})$, while the denominator the post-recombination one $(\Omega_\mathrm{k}, \Omega_\Lambda, \Omega_\mathrm{M}, H_0)$.
We better use as a free parameter $\theta_\mathrm{s}$ because it is very well measured by Planck.
What do we measure with BAO and SnIa?

- With SnIa we measure $H(z)$:

$$H(z) \propto \sqrt{\rho_{\mathrm{TOT}}(z)} \quad \text{and} \quad H_0 \propto \sqrt{\rho_{\mathrm{TOT}}(z=0)} \tag{96}$$

We have that:

$$d_{\mathrm{L}} = (1+z)\int_0^z \frac{c dz'}{H(z')} \xrightarrow[z \ll 1]{} czH_0^{-1} \simeq vH_0^{-1} \implies H(z) = \sqrt{\frac{8\pi G}{3}\sum_i \rho_i(z)} \tag{97}$$

- With BAO we can measure both the perpendicular and the parallel components of the damping scale $\theta_{\mathrm{d}}$:

$$\theta_{\mathrm{d}}^{\perp} = \frac{r_{\mathrm{s}}(z_{\mathrm{DRAG}})}{d_{\mathrm{A}}(z)}; \qquad \theta_{\mathrm{d}}^{\parallel} = r_{\mathrm{s}}(z_{\mathrm{DRAG}})H(z) \tag{98}$$

We are measuring the Hubble flow.

Why is it so difficult to solve the tension? We have that:

- At low $z$ (post recombination) we have $h \sim 0.73$
- At higher $z$ (pre-recombination) we have $h \sim 0.67$

These two values have a $5\sigma$ tension. The measure of $\theta_{\mathrm{s}}$ we are getting from CMB is model-independent and we cannot change it. Moreover, joining CMB+SnIa+BAO everything is over constrained and there is no way to solve the tension. We could do it if there was a model that predict a reduction of $7-10\%$ of the horizon size. We could play a bit with the radiation content of the early Universe going **beyod $\Lambda$CDM**:

$$\rho_{\mathrm{r}} = \left[1 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} N_{\mathrm{eff}}\right]\rho_\gamma \tag{99}$$

Usually $N_{\mathrm{eff}} = 3$ to take into account all the relativistic species at recombination. It could be interesting to see what value of $N_{\mathrm{eff}}$ could help us to obtain an $H_0$ value which matches early and late Universe measurements. However, $N_{\mathrm{eff}} \neq 3$ is not supported by Planck, which means that we should find a way to modify $r_{\mathrm{s}}$ without modifying $N_{\mathrm{eff}}$ (for example, there are ideas for early dark energy). In fact if $N_{\mathrm{eff}} > N_{\mathrm{eff,standard}}$ we have an impact on the ratio between the damping scale $\theta_{\mathrm{D}}$ and the horizon scale $\theta_{\mathrm{S}}$:

$$\frac{\theta_{\mathrm{D}}}{\theta_{\mathrm{S}}} \propto H_*^{1/2} \tag{100}$$

where $H_*$ is the Hubble parameter at recombination.

We want to understand how the cosmological parameters determine the shape of the power spectrum, so that we can understand how the CMB measurements constrain these parameters. Therefore, let us summarize all of the effects:

1. peak location $\leftrightarrow \theta_{\mathrm{S}}$
2. odd/even peak amplitude $\leftrightarrow \omega$
3. overall peak amplitude $\leftrightarrow \omega_{\mathrm{M}}$ (shifts of MR equality)
4. damping scale $\theta_{\mathrm{D}} = \frac{r_{\mathrm{D}*} \leftrightarrow \omega_{\mathrm{M}}, \omega_{\mathrm{b}}}{d_{\mathrm{A}*} \leftrightarrow \Omega_\Lambda, \omega_{\mathrm{M}}}$
5. global amplitude $\leftrightarrow A_{\mathrm{S}}$
6. global tilt $\leftrightarrow n_{\mathrm{S}}$

7. amplitude for $l < 10 \leftrightarrow \tau$
8. tilt for $l < 20 \leftrightarrow \Omega_\Lambda$ (late ISW effect)

As a side note, there is another parameter that has a tension which is $\sigma_8$ which tells how matter is clusterized and apparently we obtain a $\sigma_8$ values which predicts that matter should more clusterized than what we actually observe.

**SUMMARY AND KEY POINTS**

Before decoupling, photons and baryons can be treated as a single tightly coupled **photon-baryon fluid**. The density contrast of the photon-baryon fluid evolves as:

$$\delta_\gamma'' + \frac{R'}{1+R}\delta_\gamma' - \frac{1}{3(1+R)}\nabla^2\delta_\gamma = \frac{4}{3}\nabla^2\Psi + 4\phi'' + \frac{4R'}{1+R}\phi'$$

where this is the equation of a **harmonic oscillator** with a gravitational driving force. This oscillator equation plays an important role in the physics of the CMB anisotropies.
We can study the physics of the CMB anisotropies. The important equation is the following:

$$\frac{\delta T}{T}(\hat{n}) = \underbrace{\left(\frac{\delta_\gamma}{4} + \Psi\right)_{\eta_*}}_{\text{Sachs-Wolf}} - \underbrace{(\hat{n}v_\text{b})_{\eta_*}}_{\text{Doppler}} + \underbrace{\int_{\eta_*}^{\eta_0} d\eta(\phi' + \Psi')}_{\text{integrated Sachs-Wolf}}$$

which relates the observed temperature fluctuations in a particular direction and the corresponding fluctuations on the surface of last scattering.
The temperature anisotropies can be written as a sum over spherical harmonics:

$$\frac{\delta T(\hat{n})}{\bar{T}_0} = \sum_{l=2}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\hat{n})$$

At last, we have the relation between the parameters of the **ΛCDM model** and the shape of the CMB power spectrum. Three length scales are imprinted in the spectrum:

- The **sound horizon** at last scattering, on which the peak locations depend
- The damping of the spectrum which is determined by the **diffusion scale**
- The **distance to last scattering** on which the angular scales depend

The sound horizon and the diffusion scales are fixed by pre-recombination physics and hence in the ΛCDM model depend only on $\omega_\text{b}$ and $\omega_\text{m}$. These parameters also determine the peak heights. The angular diameter distance to the last scattering is sensitive to the geometry $(\Omega_\text{k}, H_0)$ and the energy content $(\Omega_\text{m}, \Omega_\Lambda)$ of the late Universe.