General Relativity

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i. Introduction

The real revolution performed by Einstein is to consider space and time as dynamical quantities. Geodesic structure means that free fall and measurements can be related to the same concept linking them: the **equivalence principle**. Metric is a genuinely geometrical concept, but thanks to GR it acquire physical meaning.
The issue these days is that we do not know whether the equivalence principle is working at quantum level. If it is then also GR works for quantum physics. On the other hand, if it is violated at quantum level this will open a lot of possibilities to understand gravity at quantum level. For example, if we consider a neutrino traveling in space and during its travel it changes flavor this constitutes a violation of the equivalence principle.

General Relativity

General Relativity is a theory of gravity, formulated by Einstein between 1907 and 1915, where the spacetime geometry is dealt with the same standards of dynamical variables. The physical basis of the theory is the equivalence principle.

Special relativity was related to problems connected with Galileo transformations and Maxwell's equations. GR is related to the problem of extended physics to any accelerated frames. According to this the change of the approach is connected to Einstein proposal to **extend physics to any reference frame**.
All the physics we know is formulated for inertial frames only and there is a tautology: classical and quantum physics are working when we have flat spacetime and inertial frames. But what is the inertial frame? The one in which standard mechanics is working. This problem was pointed out by Mach: we cannot formulate a theory which works only in a preferred set of reference frames.

The epoch of GR is 1915, considered an *annus mirabilis*. The contributions also go to Emmy Noether and David Hilbert. Noether realized that Einstein equations could be related to conservation laws. Hilbert formulated the field equations considering a minimal action principle. At the beginnign GR was considered only a mathematical theory, but after 1919 Eddington proved it could be tested. Now GR is indeed a physical theory!

ii. Conception of space and time before Einstein

From one side Einstein understood that space and time are physical quantities not only geometrical ones. From the other side space and time can be relative to an observer. According to these two intuitions Einstein realized that he could formulate a theory which output was space and time themselves.

Einstein's conception of space and time can be considered heretical:

- **Newton** considered space and time as absolute. They stay there without possibility to infer about them. Space and time are the sensorium dei, they are given. Immanuel Kant formalized this point: space and time are some kind of noumena, something we can be understood by intuition but on which we cannot operate. Space and time according to Newton are just an arena on which we can develop phenomena. It's useless to speculate about them as they are given. It's a very theological picture. In this arena we can perform Galilean transformations: we can add up velocities and get compositions of movements. There was no speed limit.
- **Leibniz** had a very modern approach to space and time. In some sense he gave a first relativistic point of view: space is considered the place where we have coexistent things and time is the fact we have the order of subsequent things. **The point is that without observer it is useless to discuss about space and time**. For Newton space and time are defined a priori. For Leibniz they can be considered only according to an observer. However, Newtonian conception won.

- What does it mean geometry? It means relations among objects. The debate was if there is an absolute geometry working for any situation or are there several geometries. For 2500 years the geometry was the Euclidean one made of axioms, postulates and theorems. We can establish relations between objects and the main output is the Pythagorean theorem which gives us the possibility to measure the space. We can fix the length of two points given a reference frame: the length is the same if we change reference frame! Other geometries are just some speculations. The physical space was considered as endowed with Euclidean geometry, the true geometry (the others are just matter of mathematics).
- The change of the point of view is due to **Gauss**, who realized that we can have curved structures that are intrinsically curved. We can have both positively and negatively curved space which can be considered a measure of a given manifold. Thanks to the theorema egregium, Gauss realized that manifolds can be classified under the standard of curvature.
- Anothe important point is connected with **Riemann** who defined the so called differential geometry, an important achievement for GR. Sixty years before GR, Riemann realized that the distribution of celestial object could be connected to the structure of space and time. His intuition is due to the fact that a distribution of objects could affect space and time.

The starting point of GR is the extension of special relativity. Einstein tries to overcome the problem of the conception of space and time and states that, without any metaphysical reference, space and time are something we can measure. **Measurements define space and time**. If we have a light ray traveling between two points, we can use it as a sort of rod to measure distances and times. We are not posing the problem of an absolute concept, or that it is related to an observer.
This kind of conception comes from EM interactions. *Why EM interactions are fundamental to formulate a new conception of space and time? Why do we have to put a limit to interactions? Why the speed of light is constant?* **The speed of light is an intrinsic property of space and time**. In the Maxwell theory the speed of light is related to the permittivity constant of the magnetic field and the dielectric constant of the electric field:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{1}$$

ϵ_0 and μ_0 can be exactly derived form the structure of the vacuum.
It's not just a limit imposed by hand, but an intrinsic property of spacetime. We cannot have an interaction faster than the speed of light because the speed of light is giving us the structure of spacetime which in turn is related to propagation. In order for two points to be causally connected at maximum we can connect them with the speed of light. Inside the Minkowsky cone we are causally connected. The maximum interaction is endowed with the structure of spacetime, which is the MInkowski spacetime, the one in agreement with Maxwell's equations.

III. Gravity and acceleration

What when the reference frame is not an inertial one? What is the role of acceleration? Special Relativity is in some sense Newtonian, it is related to inertial frames. Inertial reference frames are just a very special class of reference frames.
The main issue we have in dynamics is force (cause) and acceleration (effect). Einstein starts to reformulate this approach: space and time are not static and given but can be related to some kind of dynamics. How to endow spacetime with dynamics? The extraordinary intuition from which GR started is that **the effects of acceleration are indistinguishable from the effects of gravity**.
A traveler far away from any gravitational field moves along a straight line, while a traveler nearby a gravitational source starts to accelerate. Therefore, besides the Galilean equivalence principle for which gravitational and inertial mass are the same, very likely this can be connected also to acceleration. If we have a gravitational field we have acceleration, otherwise we do not have acceleration.
Einstein does not interpret the gravitational field as due to an effective force field, but as due to the curvature of spacetime, affected by the presence of a mass. Thanks to the equivalence principle not only m_I is equivalent to m_G,

but an accelerated behavior of a curved can be rectified to recover a straight lines: equivalence between curves! **We have to establish an equivalence between geodesics**. A geodesic is the shortest curve we can have between two points, which is not a straight line on a curved manifold.
Particles in a gravitational field are moving along geodesics, which are not strictly related to straight lines, but are related to the concept of minimal line between two points. The EP in the Einstein formulation just takes care of the fact that geodesics can be any not only straight. **The gravitational force does not exist, what exists is only the curvature of spacetime**.

IV. The equivalence principle

If we understand the equivalence principle then we understand GR. GR is not a closed theory. A naive definition considers the motion of a particle in a constant gravitational field. Taking into account Newtonian mechanics the EOM is given by:

$$m_I \frac{d^2\vec{x}}{dt^2} = m_G\vec{g} + \sum_k \vec{F}_k \tag{2}$$

where we have acceleration, a field related to the gravitational field and some forces acting on the particle. If we take the trajectory in an accelerated field and make the coordinate transformation in Newtonian mechanics:

$$\vec{x}' = \vec{x} - \frac{1}{2}\vec{g}t^2, \qquad t' = t \tag{3}$$

In this primed reference system:

$$m_I \left[\frac{d^2\vec{x}'}{dt'^2} + \vec{g}\right] = m_G\vec{g} + \sum_k \vec{F}_k \tag{4}$$

We can enclose the effect of gravitational field into a definition of coordinates. By a change of coordinates we can give the Newtonian equation as formulated in an inertial frame:

$$m_I \frac{d^2\vec{x}'}{dt'^2} = \sum_k \vec{F}_k \tag{5}$$

The fact that we have an inertial frame or an accelerated frame can be eliminated taking into account a suitable change of coordinates. A coordinate transformation is able to recast the standard mechanics. The main point is that we can do this because of Galilean EP: $m_I = m_G$. Starting from this static concept we can restore the standard Newtonian mechanics.
Can we do this in an electrostatic field? Inertial masses can be distinguished from charges (Millikan experiment). Other fields have proper charges, but gravitational field has no proper charges. Several people say that gravity is not a true interaction, but an induced interaction. The charge cannot be defined because it is in any case the inertial mass.

An observer O in a gravitational field feels the same laws of physics of an observer O' which is not in a gravitational field. This statement says that locally the physics must be the same despite the fact that we are in an accelerated or in an inertial frame. It is extremely important because it is connected to the universality: the laws of physics must be the same all other the Universe. We can erase the effects of gravitational fields! We can say that the gravitational force can be erased with a suitable change of coordinates: in some sense we can restore the physics we have in an inertial frame despite the presence of a gravitational field. This is the basis of GR.

The mathematical way to test EP is that if we are moving along a straight line we can say that there is no gravitational field. If we are moving along a curved line we can say that there is a gravitational field even though locally, point by point, the gravitational field can be erased. If we have a continuous curve which is derivable we can rectify locally the curve: locally the gravitational field can be erased by a coordinate transformation.

The main point of gravity is that, given the same initial velocity, all bodies follow the same trajectory in a gravitational field regardless of their internal constitution. Consider two isotopes (same atomic number, differ only for mass), for example of rubidium ^{85}Rb and ^{87}Rb. If we allow them to fall down in a void tube ($\sim$ 20m) in exactly the same way (we measure the free fall with an atomic microscope), if their trajectory is the same we can state the EP is working. The experiment says they arrive at ground in different times. Is this true? Is this connected to a violation of EP? Yes and no. If we are considering an atom we are considering a moving wave packet, which is not properly a particle. People found out that there is a difference! This could be an intrinsic difference related to a violation of EP, or the fact that these are quantum objects not properly particles can give rise a problem. The difference can be related to the fact that when dealing with an atom we are dealing with the Schroedinger equation which is not covariant! So if this is a true difference we are violating the EP at the quantum level. If atoms fall down the same way this could be a confirmation that EP is working. QM is a non local theory, while GR is a local theory (for any point we can define velocity and position). Because of the indetermination principle we cannot formulate QM as a local theory.

Another example comes from astrophysical neutrinos. Imagine that during their flight neutrinos are changing their flavor so that we can distinguish between different neutrinos. We can do this when neutrinos are moving along geodesics. According to EP, this should not be possible, because neutrinos should be completely indistinguishable. The gravitational field is inducing a change of flavor?

We can formulate the EP in two ways:

- **The strong EP**
- **The weak EP**

The weak formulation (Einstein EP) only refer to free fall: here all the laws of physics are the same locally. If we are far away from a mass asymptotically we have flat spacetime (Minkowski). Can we do this for all the other interactions? Can we consider all the theories under the same standard? In string formulation yes.
From a numerical point of view if EP is working there is always the possibility to reduce our approach to the Minkowski spacetime, which can be recovered locally. Special relativity means specilize GR to Minkowski spacetime and inertial reference frames. As soon as we can derive locally inertial coordinates we can apply the EP.
We need to investigate the EP to:

- discriminate between theories of gravity
- validity at classical and quantum level: if EP can be formulated also at quantum level, it is saying that starting form GR we can formulate a theory of quantum gravity. The core is related to the EP at quantum level. In the path integral formulation of quantum mechanics the trajectories are not geodesics: we cannot ask that particles move along geodesics. It is important to understand which is the working formulation of QM.
- if EP is working then the causal structure and the metric structure of spacetime coincide. If EP is violated they do not coincide!

The EP is the physical foundation of any metric theory of gravity! As soon as we define a metric, we can say that $m_I = m_G$ for any object, which is the WEP. If WEP and EEP are valid then physics is universal: the EP is extremely important to fix the laws of physics, it is not just related to gravity.
Local gauge means we are working in a small size and free-falling laboratory. Spacetime is endowed with a metric and the world lines are related to the metric. If they are connected then the EP is working!

The core of GR is the EP, which can be related to the fact that we do or not have an equivalence between inertial and gravitational mass.

II. Lesson 2

i. Testing equivalence principle

GR has at its foundation the EP which can be formulated in several ways: SEP, WEP and Einstein formulation involving beside the equivalence between inertial and gravitational mass, the possibility to erase the effects of acceleration as soon as we are considering geodesic motion. The main point is how to test the equivalence principle and what mathematics we need to do so.

The realization of our experiment is strictly related to the fact that we want to reproduce special relativity results. We want to erase any curvature effects. Gravitational interaction strictly depends on the curvature of spacetime. The postulates of any metric theory of gravity have to be satisfied:

- we are erasing the curvature effects thanks to the fact that geodesic structure is related to the metric structure of our physics (spacetime is endowed with a metric)
- we need something by which accelerations are set to zero: we are rectifying the geodesics, which is possible if and only if EP is working. If we are moving along geodesics locally we can erase the effect of the gravitational field. Geodesics and world lines are strictly connected to each other.
- in local free fall we have to restore special relativity laws: the gravitational field has a role only in strong curvature regime otherwise we can erase its effects

One of the main tests of EP has been proposed by Pound and Rebka in 1960. They considered the free fall of iron atom from a tower at Harward University and discovered that there is a gravitational redshift, due to the fact that during the free fall a gravitational field is present. They were able to calculate the shift of the iron emission lines and this shift is related to the strength of the gravitational field. The experiment, in fact, was performed at the top and at the ground of the tower.
They demonstrated that the gravitational interaction can be excluded from the WEP and from the Einstein EP. This means that the effects can be related with the curvature, but they are not capable of affecting the output of a non gravitational experiment.
There is the statement that also the gravitational field gravitates: gravitational effects can affect an experiment. We have also another issue connected to the fact GR is not the only theory of gravity, but it has some shortcomings. One is related to presence of dark energy and dark matter. Maybe GR can be extended at IR scales and GR is just a particular case of a class of theories. Nevertheless, all these theories have to compared with the EP. It is a discrimination among theories.
Imagine the gravitational filed has more dofs than those predicted by GR. According to this we must require that SEP must work and not the WEP, because SEP is also considering the budget related to further dofs inside dynamics. According to this the validity of SEP could be a very important way to discriminate among different theories.
Suppose we discover particles explaining dark matter. Then GR is working well. We do not need further dofs. In order to consider the SEP:

- WEP is valid for self-gravitating bodies as well as for test bodies
- the outcome of any local experiment is independent of the velocity of the free falling apparatus: special relativity point by point is valid (we have a differentiable manifold and point by point we can define a tangent space)
- the outcome of any local test experiment is independent of where and when it is performed: universality of the laws of physics (we do not actually know). If SEP is working, we can ask for the universality of physics. People say that we actually have average laws of physics. In QM we can get the response of a given apparatus only if can get an average of the measurements: we are asking for the Erenfest theorem: the average of quantum measurements coincides with the classical output of our experiment.

SEP contains the Einstein EP when gravitational forces are neglected: we don't care about gravitational interactions and results of any experiments are valid all over the Universe. On the other hand, if we do care for gravitational

forces we have to understand what kind of situation we are considering: nearby or far away form a BH. If we are considering SEP we can ask that nearby and far away a BH a non gravitational experiment gives the same results. If we are considering the gravitational interaction we can say that experiments nearby a BH could be affected by gravitational interactions: we are losing the universality of the laws of physics.
Some authors claim that the only theory coherent with SEP is GR, since it is the only theory we can do not take care of gravitational interactions. If we consider further dofs we have to take care of further interactions.

Neutrino oscillations form a strictly QM pov can be considered intrinsic property of particles. They can be generated by some transformations. Some people say that oscillations can be induced by an external field when neutrinos are traveling through it. We can take for example an electron, if it is moving in empty space its trajectory is just a straight line, but if it is moving in a magnetic field its trajectory is affected by the Lorentz force. The fact that we have a gravitational field when a neutrino is moving along a geodesic this could induce a change of flavor. If we have this, this is a strong violation of the EP. As soon as we can distinguish in a bundle of traveling neutrinos, we are violating the EP which states that particle are indistinguishable if they are moving along a given bundle of geodesics. In this case EP could be tested just at quantum level.

The EP can be tested in two ways:

1. testing the foundations of GR: EP holds for any theory of gravity or not?
2. testing the metric theories where spacetime is endowed with a metric tensor: if we have two falling atoms, they could be related to the fact if we are capable of distinguishing atoms of the same isotope the we are violating the EP. Do we have isometries or not?

The Rb experiment has been done also with stronzium: it has two isotopes, ^{37}Sr and ^{38}Sr, which are one boson and one fermion. So if we are able to distinguish a boson particle falling down from a fermion particle down, we can have a violation of the CPT theorem (if we consider the product of charge, parity and time ten this is invariant). EP can really be connected to fundamental physics. GR is not the final theory!
We have to deal with two sets of equations:

1. We can take into account also non minimal couplings: the coupling constants are not constants but are functions (gravitational interaction could change with length -Brans & Dicke theory-). If gravitational interaction depends on scale then it is strictly related to the Mach principle: the length of interaction is strictly related to any inertial frame (the butterfly effect!). Interactions are well related. Several theories state that gravitational interaction is not just a constant connected to Cavendish-like experiments, but it can depend on length. The first class of experiments testing the equivalence principle assume there could be some interactions that could give rise to non minimal coupling with matter and other non gravitational fields
2. Evolution of a gravitational field: imagine we are nearby a BH and we are going far away, the gravitational field can indeed change.

Several effective theories of physics require scalar fields. If we have scalar fields we can have non minimal coupling related to GR. The main reason to deal with scalar fields is related to the Higgs boson, which is a scalar field giving the mass dressing of particles. The existence of multiple scalar fields is related to the validity of the EP. The introduction of a scalar field can give rise to possible violations of Einstein EP. There are several theories of gravity where we do not need the validity of the EP in order to be in agreement with the dynamics of these further scalar fields.
This is considered the so called *fifth force*: the fact that we are considering enlarged theories of gravity, in the weak field limit gives rise to a feature, that the Newtonian potential can be corrected with Yukawa-like corrections. These corrections have an interaction scale λ and a strength scale α. This can be used to explain dark matter. If we expand the potential in series we can get the rotation curves. If EP is violated we have to revise theories of gravity in relation to other theories of physics.

II. Manifolds and tangent spaces

What mathematics do we need? The first concept is that of a manifold, a topological space whose points can be labeled by coordinates. Sometime they can be defined by some global properties, as the sphere, paraboloid, hyperboloid (regular rotation manifolds, obtained form the rotation of a conic curve).
An hyperboloid defines a Lorentzian manifold, meaning that time and space have different signatures!

In general, we can build an atlas patching together two representations of a sphere. It is what is done when drawing a geographic chart. We project a manifold according to a coordinate frame constituted by a set of open charts. **The concept of an open chart is the mathematical formulation of a local reference frame**. In principle, we do no know the global structure of our topological manifold (topology means we know globally the structure).
Therefore we consider pen charts: if we are considering an open set and we are capable of transforming it into another chart, the open set can give us a transition function (a way to pass form one to another set of coordinates). The transition function is what is allowing us physics: it gives us the transformation from a given reference frame to another reference frame.

This allows us to define a differentiable structure. **Mathematically GR is a theory invariant under diffeomorphisms, i.e. derivable transformations**, invariant under $GL(4)$, the linear group transformations. *Why do we need differentiable structures in order to define a manifold?* Differentiable means the possibility to have derivatives, so we can have velocities and accelerations.
But the main property we can have in spacetime is that **spacetime is by definition a continuum**. In order to be sure that our transformations are continuous what we need is differentiability. This is indeed the biggest obstacle in order to formulate a theory of quantum gravity: if require differentiability we cannot define a theory of quantum gravity as it requires a minimal length for space and time. We should confront with the indetermination principle, we cannot say a priori that spacetime is a continuum, it is a sort of quantum foam with any component of the Plank length. We lose the differentiability.
The differential structure is a property we need to have continuous transformations in the spacetime. Differentiability also gives us the possibility to write any transformation as a function. So we can recover differentiable functions. Given a transformation we can also define the reverse transformation: we can perform also the back motion, we are sure that spacetime is a continuum, where we can define derivable fields like the EM field.

Our manifold is spacetime and its differential structure gives rise to transformations that can be reversed. It's a sort of consistency condition so that our transformations are workable. This is a very important property for tensors (object that linearly transform between reference frames). The Jacobian allows us to pass from one reference frame to another.
Given a differentiable manifold and a curved spacetime, the fact that we have a differentiable manifold gives us the opportunity to define point by point a **tangent space**. We can pass between tangent spaces in a continuum and differentiable way. In a ny tangent space we can define our physics.
Therefore, a manifold is a bundle of tangent spaces and in these tangent spaces we can define the same physics. A tangent vector is a first order differential operator, defined on a tangent space of a given manifold. The tangent spaces are equivalent among each other thanks to some change of coordinates.

III. Parallel transport and covariant derivatives

It's the fact that we have to define the change of reference frame for a given point. We need to preserve physics! *What does it mean from a geometrical point of view to preserve a field?* Imagine to have an EM field, what does it mean having a parallel transport? We can define angles everywhere. In order to perform a transformation on an electric field (we want to know its value in some point), we need that it is defined in terms of strength position and verse. We need to preserve angles and we need equipollence (a vector is equipolllence). Parallel transport means

that the structure of the field should remain invariant under any transformation.
In flat space it is very easy since angles are the same and we are sure they are preserved when moving form one point to another. In a curved spacetime this is not so simple: we have to be very careful that reciprocal angles are preserved. We need a general definition of parallel transportation, alongside that of differentiable manifold. We need to define **covariant derivatives**! A covariant derivative is a definition of derivative by which we control point by point that reciprocal angles between field are preserved.

Parallel transport should be preserved if we are able to control the curvature of our motion. If we move on a sphere, from pole to equator, we need to control the changes related to curvature. We need a notion of parallel transport which takes care of the fact that geodesics are changing their angles. The parallel transport of a vector depends on the path along which it is transported. How to control it? We have to be sure that point by point respective angles are preserved. In the Frenet-Serret reference frame (velocity acceleration and binormal vector), if we have a point moving along a curve we have a natural reference frame.
If the reciprocal position of $\vec{U}$ and $\vec{V}$ is preserved we are doing parallel transport. This will translate to the geodesic equation, which is called **auto-parallel curve**, the curve in which it is possible the parallel transportation which is turn the minimal curve joining two points.
Covariant derivative means that if we are moving along a given path, this path is that our point is preserving angles of vectors connected to the moving point.
Imagine to transport a vector along a loop, the direction of the vector strictly depend on the path. The angle is saying that our manifold is curved. **The curvature is the amount of angles we can sum up when moving along a closed path**.

iii.1 Examples

A sphere is a curved manifold because the amount of our transportation is larger than π.
We have an analogue situation with a manifold with Lorentzian curvature. In Euclidean coordinates:

$$-x_0^2 + x_1^2 + x_2^2 = 1 \tag{6}$$

We want to obtain the induced metric and so we parametrise the hyperboloid. What is the length of a curve on this manifold? It is given by the integral of the velocities we are performing:

$$l = \int v dt \tag{7}$$

The length can be considered as a sort of Lagrangian. Geodesic are nothing but the Euler-Lagrange equations! Geodesics can be related to derivatives of the metric. Connections can transform geodesics from a reference frame to another reference frame.

III. Lesson 3

i. Connection

A connection is the possibility to connect two observers from one side, and from the other side it is trying to restore the physics that we have in special relativity. We have to retain the rules of derivation in order for the differential structure to work. A manifold is a collection of all tangent spaces (a tangent bundle). This is the possibility that any observer is able to reproduce the standard physics. Point by point the observer can get the standard Minkowskian physics.
The main issue in this operation is related to the basis: basis means that we can restore a sort of frame of unitary vectors on which to decompose any other vector. In order for the parallel transport to preserve the characteristics of any vector we have to be sure that moving from one tangent space to another the main properties of vectors are preserved. We have to preserve universality of fields. In order to do this we have to take into account a correction, which is the connection, which is related to the angles the vectors get rotated when moving from one reference frame to another.

We have two kind of tensors: **torsion tensor** and **curvature tensor**.
Imagine to perform a rotation in a given space: if we are not capable of putting the vector in the same point of the starting vector, this is considered torsion tensor. Imagine to move on a table with some defects: if we are not capable of going back to the same point there is some torsion. On the other hand if are rotating a vector and it does not give rise to same direction wrt the starting vector, then we are dealing with curvature.
Curvature and torsion define the inner structure of spacetime. Torsion is the commutation of first derivatives, while curvature is the commutation between second derivatives. If the connection is symmetric we have no torsion. If the space is flat there is no curvature.

How can we define the standard manifold of GR? We have that the equivalence principle is just saying that we are preserving isometries: metric is not changing in the manifold, and the covariant derivatives of the metrics are zero. Moreover, the space is torsionless, which means that the affine connection is symmetric. The general manifold has torsionless metric where we can preserve isometries. **These are Riemannian manifold with a couple: the manifold and the metric on it**. Generalizing, if we consider particles with spin and spin is contributing to dynamics then we have to consider torsion (these are called Einstein-Cartan theories, i.e. GR with torsion). We can also have metric affine theories.
GR is a theory where we have isometries, torsionless, the derivative of metric is zero and the connection is symmetric. The EP guarantees torsionless and isometries. So a rigorous statement of EP is:

Equivalence Principle

At any even of spacetime we can define a reference frame where the Levi-Civita connection vanishes at that point. Such a frame is provided by the harmonic or locally inertial coordinates and it is such that the gravitational field is locally removed. Yet the gradient of the gravitational field cannot be removed (curvature can never be removed because it is a tensor).

Up till now we have given a physical formulation of the EP, which means that beside asking $m_I = m_G$ we also require the existence of a reference frame where we can remove the acceleration. The free fall is the situation where point by point we can remove acceleration.
From a mathematical point of view, it means that we can define an event (a point in spacetime) in which we can define a locally inertial coordinate system which allows to vanish the Γ but not the curvature.
An harmonic coordinate system means that along a curve we can have a vector: the harmonic coordinates are the corresponding vector to the curve point by point. They are considered harmonic because the Schwarzschild rule holds: the commutation of second derivatives is zero. The final output is that the geodesic equation can be reduced

to:

$$\frac{d^2\xi}{dt^2} = 0 \tag{8}$$

So geodesics are straight lines. The physics we have in special relativity is the same in any reference frame if EP is working. **The structure of spacetime can be defined by the properties of transport**. Transportation gives all the structure we need.

II. Gravity and geometry

Gravity is the prototype of any theory that can be dealt under the standard of geometry. We have to fix the following analogies:

- an event is a point labeled with a time and a position, in geometry it is an element of a topological space which is the manifold defining our spacetime.
- spacetime in physics is the set of all events that give rise to a continuum (we need differentiation); in geometry it is a differentiable manifold
- in physics gravitational phenomena are manifestations of geometry of spacetime; in geometry it is the metric g on a manifold
- particles move in spacetime following special world-lines, which are straight lines (i.e. geodesics)
- laws of physics are the same for all observers, because any observer can restore flat space time on a given point
- field equations are covarinat under diffeomorphisms

This is a geometrization of physics. In the other fields we are always distinguishing mathematical tools and physical observation. In this case field equations are nothing else than transformations. **Geometry coincides with physics!** According to Einstein the mathematical model of spacetime is a pair: a differentiable manifold and a metric.

There is also the Palatini formulation (1919): if we are considering connections not related with metric we can deal with gravity under the same standard of Hamiltonian formulation. Einstein liked this idea and tried to formulate GR accordingly. Palatini approach means metric affine approach, i.e. connections and metric can be disentangled, not exactly given by the Levi-Civita connections. In GR however connection and metric coincide. The metric affine formulation is more general.

Metric formulation means that dynamics is fully endowed in the metric. Metric affine formulation means that dynamics is endowed in metrics and in connections. Metric affine formulation is more coherent with the Hamiltonian formulation where position and momenta can be different. On the other hand, in the Lagrangian formulation velocities come directly from positions. If we want a Hamiltonian formulation of GR and then a possible quantization path we have to take into account the metric affine formulation.
In the Palatini approach[1] Christoffel symbols could be different wrt the Levi-Civita connections. We can have that a combination of metric and its derivatives give rise to the connection.

At the moment gravity is not considered a fundamental interaction because in order to have it we need a quantum formulation of the theory. Is gravity a true interaction or an induced one?

So we have that:

[1] Gravity trinity approach: we can get GR in several ways, starting from metric, starting from torsion or starting from non metricity. All these formulations are connected for something. If take into account torsion other curvature we do not require a geodesic structure (tele-parallel equivalent gravity). We have affinities and we do not need the EP. In the metric approach we need strictly that the EP has to be retained. That brings to the Levi-Civita connection. In a metric affine situation the Γ are not Levi-Civita and the free fall is another metric structure. Non metricity means that changing reference frame the covariante derivative of metric is different from zero. This could be interesting for gauge theories. Isometries mean we are imposing a gauge. In this gauge we allow the gravitational field to change along the manifold

- differentiable manifold = spacetime
- metric = gravitational potential
- connections = forces

If the EP works there is a reference frame where we can set the connections to zero. There is also a strong statement saying that matter and geometry are the same thing: we can produce particles from geometry and geometry can be produced by particles. It is likely connected to a very deep formulation of quantum gravity.
The main point however is if gravity can be connected with the other field or not since the structure of spacetime is something very complicated. The differential structure we are asking for implies that time is a continuum. In QM when we write the commutation relations we need to now a priori the form of spacetime, but this is the output in GR. If we consider two metrics in two points and write the commutation relation, we get something, but the shortcoming is that g is defining the metric but we need the metric to define spacetime. Formulating commutation rules for gravity is misleading. At quantum level space and time cannot be defined in a straightforward way, although they are good concepts at classical level because we can consider differentiation and continuity.

III. Geodesic equation

If have that the geodesic equation is written as $\frac{d^2\xi}{dt^2} = 0$ then the metric is the standard Minkowski metric. Let us now change our coordinates and pass from ξ coordinates to x coordinates. Passing from one reference frame to another means to be regular in the definition of Jacobian matrices. If they are different from zero we can pass from the standard Minkowski metric to another metric. The possibility to pass from one reference frame to another is just a matter of differentiation, of diffeomorphisms. So if the Jacobian matrix is different from zero we can define any metric in any curved space.
The connection is nothing else but the second derivatives of the change of coordinates. Starting from straight lines and arriving to any lines is just a matter of change of reference frame but we need regular diffeomorphisms. We need the Jacobian matrix to be different from zero. Moving between two reference frames is a mater of change of coordinates. Gravitational force means the possibility to curve some geodesics.
The analogy with the Newtonian force is just that the affine connections are the generalization of the Newtonian gravitational field and the metric tensor is just the generalization of the Newtonian gravitational potential. The difference is that in Newtonian formulation we have just one potential, while in GR we have 10 potentials giving rise to the gravitational field. GR give s a richer dynamics. The notion of metric and connections emerge in a natural way and describe the gravitational fields as soon as we change reference frame and describe falling bodies. However, all this stuff is working only if the EP is working: the EP has both a geometric and physical role. The geometrical meaning is that every bundle, straight and curved, coincide, while the physical meaning is that assuming $m_I = m_G$ we can erase the effect of acceleration in a given frame. This also mean that the force cannot be an absolute concept.

Why do we need a tensor to describe a gravitational filed? Because forces are not an absolute concept. However, we have a big issue: **what is a connection?** It must be related with the metric. If we are able to relate metric and connection we can say that this is the final output of the equivalence principle. Up to now we have seen that connection comes out of a change of coordinates. Now we want to show the relation between Levi-Civita connection and the metric derivatives strictly giving rise to the fact that force derive from the gradient of potential.

IV. Connections and metric

Let's start form the covariant derivatives of a metric. Isometries are preserved on our manifold.:

$$g_{\mu\nu;\alpha} = 0 \tag{9}$$

Being a second order tensor we need two corrections:

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} - \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} = 0 \tag{10}$$

So the standard derivative of the metric is just:

$$g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\alpha\mu} g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu} g_{\alpha\nu} \tag{11}$$

Rotating the indexes we can end up with:

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} \right) \tag{12}$$

V. The geodesic motion

If the EP is working, there is only one choice for he affine connection and it is the Levi-Civita connection. This formula is saying that spacetime is so that geodesic and metric structure are coinciding, it is saying that spacetime is isometric (in any measurement frame we can have the same measurements), it is saying that spacetime is torsionless. All these stems from the validity of the EP.
So geodesic motion is a matter of change of coordinates. If we want dynamics we need that our field equations must be the Euler-Lagrange equations coming from a variational principle. **We must demonstrate that our geodesic comes directly from a variational principle.** The variational principle is saying that metric and geodesics are connected to each other. If we want to demonstrate that the metric structure and the geometric structure are indeed the same we need to prove that the geometric structure comes out from the metrics.
We can formulate the problem as a variational problem. To choose the invariant we can simply take the metric invariant. We want to see if variating this invariant we get the geodesic equation. Remember that we use the fixed board problem. The minimization is achieved if we get the geodesic equation.
The result is that the EOM of a particle moving in free fall is the gravitational field. So from the metric we have derived the geodesic. Metric and geodesic structures coincide if the equivalence principle is working. In the Palatini formulation, for example, we cannot get the geodesic from a variational principle. Or if we have torsion the previous calculation does not hold. Considering the EP also at quantum level is extremely important in order to have a self-contained and self-consistent theory. If EP is woking at quantum level we can retain GR as we have constructed it, otherwise we have to revise all the physics.
If we want to measure using light rays (this is geodesic and also a rod to calculate distances), if geodesic and causal structure are different measurements done with light rays differ from measurements taken with a rod and a clock.

VI. Parallel transport and geodesic equation

We have to show that geodesic and parallel structure coincide. In moving along a give trajectory parallel transport means that reciprocal directions should remain at the same angles. Parallel transport is geodesic transport. **A geodesic is an auto-parallel curve: geodesics are those curves which parallel transport their own tangent vector**. Along geodesics the tangent vector remains auto-parallel. If a physical field is moving along a geodesic it remains invariant.
If we are moving in an electric field, we can have geodesics. If we are moving in an electric and magnetic field we have the Lorentz force and we cannot have geodesics (auto-parallel curves).

Summing up, the main issues are mathematical tools to formulate GR: metric, manifold and the connection on the manifold. We defined parallel transportation that can be related to geodesics. The geodesic structure can be related to a metric structure if there is a variational principle with which starting from the metric we can get the geodesic. This is allowed only if EP is working. Therefore, making measurements along geodesics is the same thing as making measurements using the standard Pythagorean theorem. Finally, a geodesic is an auto-parallel curve, where the tangent vectors are parallel transported.

IV. Lesson 4

I. Weak field limit

In order to be a reliable theory we need to recover the weak field limit. We have to recover the Newtonian mechanics, which means the Kepler laws from one side and from the other side the gravitational field related to the Newtonian potential. Our geodesic equation should reduce to standard orbits. We have to recover standard Keplerian orbits we have in celestial mechanics.

We have to specialize the geodesic equation to standard trajectories to recover the orbits. What we have is that space is distinguished from time. Moreover, the geodesic equation, which formulated in 4 dimensions has to be split in the time part and in the space part, because trajectories in standard mechanics are defined in standard space which is disentangled from the time. In doing this the metric is not time dependent. It can be considered stationary and in such a case we can say that:

$$g_{\alpha\beta,0} = 0 \tag{13}$$

We are assuming stationary gravitational fields and the four geodesics become three geodesics, because we split it in a part which gives the standard time and a part which gives space:

$$\frac{d^2x^k}{ds^2} + \Gamma^k_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{14}$$

We can evaluate the second part using the weak field limit. Weak field limit means that we are assuming that gravity is weak and velocities are much lower than c. **Which is the standard velocity we can consider in Newtonian mechanics? What does it mean a small velocity wrt the speed of light in celestial mechanics?**
Small means standard velocities we can have in our Solar System. We can take into account the proper motion of stars. The typical velocities of stars is of order of 100km/s. The Sun orbits with 200km/s. At Einstein times these velocities were very well know. Therefore, wrt the speed of light:

$$\frac{v}{c} \sim 10^{-3} \tag{15}$$

This can be considered a small velocity we can have in a standard astrophysical environment.
On the other hand, when we can say that a gravitational field is weak? We can imagine to have an object orbiting around a planet/star. In a stationary orbit the kinetic energy of our body should be equal to the gravitational energy:

$$\frac{1}{2}v^2 = \frac{GM}{r} \implies v^2 = 2\frac{GM}{r} \tag{16}$$

If we divide both terms by c^2 we get:

$$\left(\frac{v}{c}\right)^2 = \frac{2GM}{c^2 r} \tag{17}$$

This means considering the previous result:

$$\frac{2GM}{c^2 r} = \frac{\phi}{c^2} \sim 10^{-6} \tag{18}$$

So in the weak field approximation we are considering $v/c \sim 10^{-3}$ and $\phi/c^2 \sim 10^{-6}$. This is the precise definition of the weak field limit.
So we can write:

$$\Gamma^k_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \Gamma^k_{00}\left(\frac{dx^0}{ds}\right)^2 + 2\Gamma^k_{0i}\frac{dx^0}{ds}\frac{dx^i}{ds} + \Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \tag{19}$$

With the weak field approximation:

$$\Gamma^k_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \simeq \Gamma^k_{00} \tag{20}$$

Therefore the EOM becomes:

$$\frac{d^2x^k}{c^2dt^2} + \Gamma^k_{00} = 0 \tag{21}$$

Since we are in the weak field limit we can use the Minkowski spacetime $\eta_{\mu\nu}$. So:

$$\Gamma^k_{00} = \frac{1}{2}g_{00,k} \tag{22}$$

We can interpret g_{00} as a potential, because if we take into account the Newton equation, this says that the acceleration in a gravitational field is given by the gradient of the potential:

$$\frac{d^2x^k}{dt^2} = \frac{\partial U}{\partial x^k} \tag{23}$$

Flat spacetime means that asymptotically we have to recover the Minkowski spacetime. Very far form a given distribution of masses we get the standard Newtonian potential: $g_{00} \to 1$. We can cast:

$$g_{00} = 1 - \frac{R_S}{R} \tag{24}$$

Exactly on the Schwarzschild radius $g_{00} \to 0$, which means that **as soon as the size of a body reaches its Schwarzschild radius we lose the meaning of time**. We can state that as $R = R_S$ we are in a black hole, which is a system where the gravitational force is so strong that the radius of the object coincides with its Schawarzschild radius. The light cannot come out because the gravitational field is too strong, the meaning of time and the possibility to go outside the gravitational field is forbidden. We are in a singularity! Also in the weak field regime we are recovering a new feature, the existence of singularities.

II. Physical meaning of the metric

The metric is a geometric object. It is the gravitational potential. In the weak limit we only need one entry g_0 corresponding to the standard Newtonian potential. In general, we have a number of gravitational potentials corresponding to the number of entries we have in the metric. In 4 dimensions the metric has 16 entries, corresponding to the same number of gravitational potentials, but we have to take care of the fact the the metric is a symmetric object. So we can take into account 10 gravitational potentials. Among these **only 6 are independent**.

ii.1 Clock synchronization

What does it mean synchronizing clocks? Are we able to synchronize clocks in any case? No!
In special relativity we know how to synchronize clock because we know how time transforms in different reference frames. In special relativity and in classical mechanics we can in any case synchronize clocks as soon as we take into account the correct Lorentz transformations.
In GR we cannot synchronize clocks in any case, but if and only if the spacetime is homogeneous. Homogeneous means that our metrics cannot contain boost, that means terms that contain the mixing of spacetime components.

In general, we can define the proper motion of a given observer, the coordinate motion and then the proper time and the coordinate time. What is the difference? In a proper reference frame we are in the tangent space of a given point on our spacetime manifold: so we are not feeling any gravitational field, we are solidal with the reference frame, we are perceiving in this local gauge the standard physics. On the other hand, if we are a coordinate observer observing another proper observer we are also measuring the gravitational field. In order to pass from proper time to coordinate time we have to take into account the effects of gravitational field.
If we are in standard special relativity g_{00} is a Lorentz transformation and so we are able to promptly relate proper and coordinate time. If we are not in this situation, g_{00} gives the effect in passing from proper time to another coordinate time.

Gedanken experiment: we have two observers A and B that want to synchronize their clocks. Using the fact that EP is working and that a light ray can be considered a standard rod to take measurements, B can send light rays to A who has a mirror and can send back the rays. Then if we want to synchronize the clocks we must set: $x_0 = x_0^*$. x_0^* can be defined as the average position on the light cone between $x_0 + dx_0^1$ and $x_0 + dx_0^2$. We have to make an average between the starting point where we are emitting the light rays and the receiving point where we receive the light ray. In doing this we have to take into account that the light ray propagates as $ds^2 = 0$. If we develop in terms of coordinates.
If $g_{00} = 0$ (on a black hole) we cannot synchronize our clocks. If the off diagonal elements are different from zero we cannot synchronize either. g_{0k} are the spacetime rotations: if one reference frame is rotating wrt the other we lose the possibility to synchronize our clocks. **Synchronization strictly depends on rotation**! If we have a system of satellites moving around Earth we must take care of rotation. So the GPS is a web of satellites $(14-20)$ and they cover for the fact that Earth is rotating.
Rotations only appear in GR. In special relativity we are considering inertial frames. In GR synchronization is not possible if we have metric elements related with rotation. **A metric is homogeneous if we do not have off-diagonal elements**.

This is not just an effect related to GR, but it is present anytime a system a system is rotating wrt another system. Let's take an inertial frame and consider it in cylindrical coordinates and put the system in rotation: all the coordinates remain the same except for the angle $\phi \to \phi + \Omega t$. This mixes time and angle! We have $g_{0k} = -\Omega r^2/c$. If Ω is large (rotation of a pulsar for example), we have to take into account rotation and synchronization can be a very demanding issue.
This is an example of the fact that metric has a physical meaning and this is related to the fact that we are taking into account the synchronization and this can be a very demanding effect according to the fact that we have to correct the synchronizations. This is connected to the timing of pulsars. **Synchronization is possible if we know the physical structure of rotating objects!** Nearby a BH we have strong deformation of spacetime and very strong tidal forces that give rise to rotations. Physical information means we need measurements in space and time, but if we lose the possibility to get measurements we lose the possibility to get info on the system.
Any astrophysical body is rotating, but the point is that if rotation is not strong it's okay. If rotation is strong we have to take into account the Kerr metric.
In principle, we can synchronize clocks also in GR, but a posteriori because we need info on the structure of the rotating body. In special relativity we only need to know what is the velocity of an observer wrt the other. In GR we need not only the velocities but also the structure of the body (mass, velocity, radius). We need information on the rotation, it is not just a matter of a transformation. This can be problematic, because the metric outside the system is related to the fact that we need to solve the Einstein field equations, which is a delicate job.

III. Riemann tensor

It's the core of GR. Spacetime is curved and curvature corresponds to the gravitational field. Riemann in his PhD thesis stated that spacetime is shaped by the distribution of objects and the distribution of the objects curve the spacetime. **What does it mean considering a curved spacetime?** We have to compare our curvature with some invariant results. If we take a physical vector and we move it along a closed path, if the vector is not recovering the same initial position and direction and verse, the vector is changed and this is an indication of the fact the spacetime is curved.

The total variation of the vector is given by a tensor, which is the curvature tensor. If the effect of a travel along a closed path is different from zero then the manifold is intrinsically curved and the amount of curvature is given by a tensor which is the Riemann tensor. A combination of second derivative of the metric is the Riemann tensor. This is definition in terms of coordinates.

Torsion and curvature mean that we are moving our vector along a given direction and the result is that the object is not at the same direction we consider at the starting point. We can also have another definition. Imagine to have a geodesic triangle (a triangle on a sphere), the parallel transport is strictly related to the fact that if the spurious transformation is zero then the spacetime is flat. Remember the Green-Stokes theorem (the rotation of a vector correspond to the flux of the vector).
The Riemann tensor can also be defined as a commutation of second covariant derivatives. Being strictly related to commutation relations, we have to take into account in the eventuality of quantization. The Riemann tensor gives the possibility to check the inner structure of our manifold. The Riemann tensor is a commutator and it could be an intrinsic object that could help us to quantize gravity.

V. Lesson 5

I. Properties of Riemann tensor

When we defined the Riemann tensor, we got a relation that says that it is a combination of second order derivatives of the metric. So second order derivatives of the metrics is nothing else but Laplace operator applied to gravitational potentials. Since the Riemann tensor is related to derivatives of second order, it must be related to dynamics of gravitational potential.
Beside this we an important issue related to the tensor, which is the Bianchi identity. Luigi Bianchi (Pisa) found a nice relation related to cyclic derivatives of the Riemann tensor. This feature is fundamental, because it not only is an algebraic property but it is connected with conservation properties. Einstein field equations can be related to conservation laws.
The Riemann tensor is an object with several symmetries related to pairs of indexes. Petrov classified these symmetries. The number of independent components of the Riemann tensor is 20 out of a total of 256 components. Nevertheless, the number of components depends on the dimension of the space on which we are defining our Riemann tensor. In 4 dimensions we have to deal with 20 field equations, even if the number of the independent components of the metric is less than 20.

An important property of the tensor is the relation to the commutation of second derivatives. The fact that the commutation of second derivatives of a given vector field is different from zero is extremely important in order to define an algebra of commutators.
Heisenberg algebra is taking the Poisson commutations in Hamiltonian formulation. The difference is related to Heisenberg uncertainty principle.
The presence of a gravitational field is strictly related to the fact that the commutation of a given vector is different from zero when taking into account second order derivatives. According to this we have that a manifold is intrinsically curved and we are considering dynamics of the gravitational field, which is related to the Riemann tensor.
Another property of the Riemann tensor is related to contraction:

- Ricci tensor: contraction of first and third indexes. It is fully symmetric. Its symmetry is the same symmetry we have in the metric. The Ricci tensor can be strictly related to the main properties of a gravitational field since the main goal of solving Einstein equation is to find solutions which are nothing else but metrics
- Ricci scalar

Combining them we get the Einstein tensor:

$$G_{\mu\rho} = R_{\mu\rho} - \frac{1}{2} g_{\mu\rho} R \tag{25}$$

The important property of this object is that its divergence is zero:

$$\left(R_{\mu\rho} - \frac{1}{2} g_{\mu\rho} R \right)_{;\mu} = 0 \tag{26}$$

This is connected with a conservation law.

II. Einstein field equations

They are second order field equations defining the dynamics of the gravitational field. Solving them we get the potentials by which we define the gravitational field in a given region of space. They describe any gravitational phenomena.

The Einstein approach was a sort of inverse scattering approach: there are some properties by which we can define suitable field equations in relation to the gravitational field. Essentially the filed equations should have some properties:

- We have to write the field equations in agreement with the covariant principle: they should remain in the same form in any reference frame. They must be the same for the whole Universe!
- They must be second order in order to require standard dynamics. It is not a very strong requirement. The consideration is that all field equations in physics are second order because second order means dynamics and dynamics means acceleration. Maxwell's equations are second order wrt the vector potential. Or the Schroedinger equation is second order wrt the spatial part. So it's an analogy: in order to require dynamics we need at least second order. It's a minimal requirement
- In order to be consistent they should reproduce Newton laws in the weak field limit

We have already asked that form the geodesic equation we should be capable of recovering the laws of universal gravity. We supposed that the metric was given. We haven't derived the equation according to the shape of spacetime. Now we are no longer supposing a form of spacetime on which to define trajectories, but we are trying to find out equations which output is directly the spacetime. GR aims at finding out the form of spacetime. Geodesics are written on a given spacetime but they do not directly give us the spacetime. Now we require that **spacetime should be the direct result of given equations**. Recovering Newton laws means that in the weak field limit we are not just recovering the orbits but we are recovering the form of the potentials. We have to recover the Laplace and Poisson equations, which are equations giving directly the gravitational field. In some limit, something related with curvature must give us the potential and the variations of the potentials according to the distribution of matter. The meaning of GR is to directly find out spacetime. The Cauchy problem of Einstein field equations was solved in 1955 by Madame Chokebroic. She was the first to find out the Cauchy problem for the Einstein field equations. It is very complicated. Now it is a debated issue: every problem in physics must have a well defined Cauchy problem otherwise we lose predictivity.

Einstein first defined his equations in vacuum and then in presence of matter. In vacuum what defines the property of space is the Riemann tensor, so in vacuum it is zero. This is correct but not sufficient. According to the fact that Riemann tensor globally defines the properties of spacetime it means that the only possible solution is the Minkowski spacetime. So we need some combination of curvature invariants so that we have solutions in vacuum tat are not only the Minkowski spacetime.
Imagine to have a star and we are at the border. Inside the star we have to define the field equation in presence of matter. Far away from the star however we cannot strictly impose that spacetime is flat. The more if we are considering a compact object.
Instead of considering the Riemann tensor equal to zero, Einstein started to take into account the Ricci tensor equal to zero. This does not imply that the Riemann tensor equals zero. There is a problem however. Einstein defined a stress-energy tensor to describe the distribution of a perfect fluid to be equal to the Ricci tensor in presence of matter, but the stress tensor satisfies the Bianchi identity and the Ricci tensor not. The mistake is that we need two tensor which respect conservation laws (i.e. the divergence of a given tensor). The tensor capable of giving the right field equations in general (vacuum and matter) is the Einstein tensor, respecting Bianchi identities and the conservation laws.

If we consider a stationary (the metric does not depend on time) weak field, the R_{00} component of the Ricci tensor gives rise to the Laplace equation. We get also in this case the correct weak field limit.

ii.1 The cosmological constant

The Universe is evolving if we consider dynamics equations. So Einstein plugged by hand a cosmological constant in his equations to stabilize them so that the final solution of the equation is a static solution. Hubble demostrated then the Universe is a system in expansion and changes with time. Einstein then considered the cosmological constant his biggest mistake. Then the cosmological constant was again introduce to explain the expansion of the Universe (it gives rise to an effective pressure representing dark energy).

ii.2 Field equations from a variational principle

If we want the equations to be real field equations they must be derived from a variational principle. Emmy Noether realized that Einstein's (in terms of conservation laws) and Hilbert's (in terms of variations) approach were the same.

In defining a Lagrangian we need to take something which is an invariant and this invariant must be a scalar (because the Lagrangian is a scalar function and has to be universal, defined all over the manifold). In this case we also need to define a volume on which the Lagrangian strictly depends. What does it mean to have a covariant volume? This is related to the Liouville theorem, which says that flux lines have to be preserved (the number of flux lines passing through a surface must be preserved). So our volume is invariant if any transformation is giving us the same number of flux lines. Despite any transformation we can get the same form of the metric. From a mathematical point of view it means that when changing coordinates we need a Jacobian determinant. We need the same volume for any transformation. The Jacobian determinant is related with volume.

On the other hand, what is the scalar invariant that can give rise to the correct field equations? It's the Ricci curvature scalar. We obtain that the Einstein tensor coincides exactly with the Euler-Lagrange equations derived from a variational principle where variation is performed wrt the curvature invariant, i.e. the Ricci scalar. So Einstein tensor is not just a lucky chance related to Bianchi identities of the Riemann tensor, but it is also connected to the Euler-Lagrange equations coming from the Ricci scalar. This result was controversial at the time: two completely different approaches give the same result.
It was realized by Emmy Noether that if we have Euler-Lagrange equations they are real field equations if they satisfy conserved quantities. Conserved quantities mean exactly we need Euler-Lagrange equations. So any theory of gravity producing field equations, these must satisfy Bianchi identity (it's a consistency check). After this result, it was realized that also Maxwell's equations satisfy this rule. Also here we have Bianchi identities, they are conservation equations.

VI. Lesson 6

i. Stress-energy tensor

Let's consider now the field equations when we are in presence of matter, some source that is feeding the field equations. When we are supposing matter is shaping space this means we have to take into account two situations: inside and outside the body. Take a star: we have to distinguish the internal and the external space. In order to be self-consistent our problem has to be define d according to some boundary conditions that must be defined on the border of the star. Any solution has to distinguish between internal and external solution and it must be continuous and derivable in passing from internal to external part. Sometimes it is very difficult to have a good formulation for the Cauchy problem.
Madame Chokebroic achieved this solution: if we have an internal and an external solution, in order to join them we need that the boundary conditions have to be formulated both for the field equations and for the fluid which is feeding the field equations.

What is the form of the fluid that is the source for the Einstein's field equations? We'll discuss a particular fluid, which is a standard fluid which should cover all the possibilities.
If we want the equations to be consistent we need a source which is a fluid, because this fluid is consistent with fluidodynamics. Any source could be reshape in the form of a fluid. If we have a fluid that is conserved according to Bianchi identities, we can consider it as a source for Einstein field equations. Madame Chokebroic realized this: if we want that Einstein equations are formulted such that the Cauchy problem is well posed we need to understand what kind of fluid can be considered as a source. We need a source that can be reshaped as a fluid. This is a problem when we want to extend GR to cosmology as we need that this fluid is not a perfect fluid. We can have a scalar field or a spin field. But if we reshape it as a fluid immediately it satisfies the Bianchi identities and can act as a source for Einstein field equations.

GR can be defined as a sort of extended fluidodynamics, which is a relativistic one. We need to define a fluid that can act as a source. What kind of fluid can we define? It must be a covariant fluid, which should be the same in any reference frame. This fluid should be a Liouville fluid: it must respect the Liouville theorem (the number of flux lines of the fluid is preserved). If we hav eneither source nor sink, it is a conservation statement.
How to define a fluid in agreement with GR? It must be an invarinat density. We must also put the mass condition: the masses are preserved. However, we are not dealing with masses, but with densities. According to this we can define the source in terms of a tensor related to our perfect fluid:

$$T^{\alpha\beta} = \rho_{(0)} c^2 u^\alpha u^\beta \tag{27}$$

where $\rho_{(0)}$ is the proper density. We can now consider the Euler equation which says that we have to take care not only of the density of the fluid but also of the pressure, which means that particle are interacting. They can move along a flux and they must satisfy the Euler equation (gives the variation of flux lines along a flow).
The energy-momentum tensor is symmetric. The pressure is subtracted (as in Wan der Vaals equations), because we are subtracting self-interactions.
If we consider covariant derivatives of $T^{\alpha\beta}$ we satisfy the Bianchi identity! We have that both the Einstein tensor and the stress-energy tensor satisfy the Bianchi identity. So the field equations in presence of matter are written as:

$$G_{\alpha\beta} = \chi T_{\alpha\beta} \tag{28}$$

We don't care about the structure, but only about the fact that we need that a given property is satisfied, the Bianchi identity. Any fluid satisfying the Bianchi identities can be the source for Einstein field equations.
We can have another interesting relation. If we take the trace of $T_{\alpha\beta}$ we get $\epsilon - 3p$. This means that **the scalar curvature tensor is exactly the trace of the stress energy tensor: the distribution of matter gives the**

curvature! This gives us an equivalent form of the Einstein field equations:

$$R_{\alpha\beta} = \chi \left(T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T \right) \tag{29}$$

The shape of the Einstein field equations can be reversed according to the trace of the stress-energy tensor. This symmetry is very important, because it is a sort of duality we can have with respect to matter from one side and with respect to the curvature from the other side. In one case we see that the stress-energy tensor is acting as the source for the field equations, in the latter form we see that it is the curvature that is acting as the source for matter. Matter equations and geometric equations can be exactly the same thing. Matter and geometry in the framework of GR are completely equivalent. It's the first time in the history of physics that the structure of spacetime is completely governed by the form of the stress-energy tensor.
This symmetry is really relevant when constructing field theory on curved spacetime.

However, the fact that we are equating two tensor according to a property can be completely arbitrary. A key role is played by the coupling constant, which is the physical point saying if we are capable or not to equate these objects. χ is the coupling constant that has to guarantee that our theory is from one side a gravitational theory and from the other e relativistic theory.
If remember that the 00 component of the Ricci tensor is the Laplace equation, we can consider the weak field limit of our field equations. Taking into account the matter formulation of the field equations, the entries of these equations can be compared with the trace of the stress-energy tensor. In the weak field limit we have no pressure related to particles interactions (particles can be considered as dust with no interactions). We obtain the Poisson equation. Therefore, we can read the coupling constant value:

$$\chi = \frac{8\pi G}{c^4} \tag{30}$$

It's extremely important because it gives the strength of the interaction we are considering. We have G meaning it's a gravitational interaction, but it is also a relativistic interaction as we have c^4. From this we get that gravitational interaction is the weakest one. So we have formulated a relativistic theory of gravity.

What is the number of independent equations? 6. We start from the 10 independent components of the metric, minus the 4 conditions given by the Bianchi identities. So in 4 dimensions we must have 6 independent equations. But if we impose a spherical symmetry, like in Schwarzschild, we get just 2 independent equations. Moreover, a spherical symmetry implies a static and stationary solution (Birchoff theorem). **The Birchoff theorem states that if we have a spherical symmetry the solution is static and stationary, and so independent of time.**

II. Another view of Einstein equations

It's the gauge picture of the field equations. We will develop the so called viel-bein formalism. This means that instead of taking into account the manifold on which we are going to formulate our field equations, we are going to take into account the observer and in particular the observer which is tangent to a given manifold. According to these we can formulate field equations completely in the spirit of the EP. In this case, we immediately ave that the Bianchi identities are setting the consistency and the physical correctness of our equations.

Vielbein mean that we can take a locally inertial frame and we can see how this frame is transforming along a given manifold. We already discussed this when we took into account the definition of parallel transport and the definition of the manifold. A vielbein is the fact that if we take a coordinate x^μ, there is the lift operator by which we can transport the vector from the coordinates of the manifold to the coordinates of the tangent space ξ^a[2]

[2] Latin indexes mean coordinates on the tangent space, while Greek indexes mean coordinates wrt the coordinate space. The system ξ^a is an anolonomic system (does not depend on time and it is related with the tangent framework), while the system x^μ is an olonomic system (related with standard coordinates, which are the physical ones)

In this approach we can take into account other degrees of freedom like spin connection. This is called the Poincarè connection, which takes into account all the possible moves we can have in our space. In particular, we can consider translations, spin connection and Lorentz rotations, all the possible movements we can have according to the parallel transport.
We can use standard formulas for curvature and derive explicitly the expression for translation which is a sort of torsion. According to this we can construct the so called structure constants according to this Poincarè algebra. We see that so our field equations can be related to the Bianchi identities. If the field equations are respecting the Bianchi identities we can consider any degrees of freedom contributing to the dynamics. Degrees of freedom could translations, spin connection, Lorentz rotations and also torsion. Torsion means that when we are moving along a closed path we are not able to recover the starting position of our vector.
According to this we can get an equation for torsion and curvature. This is the tangent bundle approach: the dofs are related with all the possibilities we have in our movements. Any degree of freedom can be dealt under the standard of gravitation, but gravitation has to be considered as a gauge theory [*See Straumann book*].
We can derive the Einstein tensor, the Ricci tensor and the curvature scalar in the vielbein approach, which means that instead of contracting with the standard metric, we are contracting with the flat metric related to the vielbein. According to the vielbein picture, if we have a spin particle falling down along a geodesic, it is related to any other particle with the same mass, we cannot distinguish them. But if we are taking into account this approach, which is called **Poincarè gravity**, because in the vielbein representation we have spin, we can deal with further degrees if freedom.
We can combine the Einstein field equations with Bianchi identities and we can achieve some statements [slide 147]. The Einstein field equations are not just a matter of framework in which we are producing them, but they are related with the fact that there are some geometrical structure independent of the framework.
We can also have the action principle with the vielbein. In the vielbein representation we can consider any form of matter. Spin cannot be framed in GR. In order to consider it we have to see it as a part of transformations. Spin is a proper length we are introducing in our dynamics. This proper length is related to an intrinsic angular momentum which is the spin. Any particle does not just bring mass, but also a length. When we are performing a loop on a given manifold, we cannot recover the starting point due to the presence of such length. According to this, we can state that our particle is also intrinsically length. Therefore, we must also consider torsion in our dynamics. **In the classical view of GR torsion can be discarded because particles are spinless**. To take care of all these degrees of freedom we need the Poincarè representation of gravity which can be achieved in a more general gauge space.

III. Solving the Einstein field equations

It means a lot of things, in particular to find out solutions according to which such solutions satisfy both the Einstein equations and the Bianchi identities. Being non linear partial derivative equations, we have theorems of existence but not of unicity[3]. We have an infinite number of solutions. How to construct a solution?
It can be related to the fact that we can find a particular vielbein, which means taking into account that the solution is spherical. We are constructing the so called Schwarzschild solution. If the entries of our field equations are spherical solutions then the gravitational potential should go as $1/r$ as in standard Newtonian mechanics. The boundary condition for asymptotic flatness means that at infinity we get zero. The Schwarzschild solution is consistent with asymptotic flatness.

Let's consider the corresponding Lagrangian. We can find the Euler-Lagrange equations. The Langrangian can give us the gammas. We have first integral of motion: energy and conservation of angular momentum. These are conserved quantities of our field equations.
The Schwarzschild is the most general static and spherically solution.

[3] Linear differential equations have theorems of both existence and unicity

Constructing a solution means to find out a particular symmetry and to impose it our spacetime. The symmetry gives the possibility to construct a particular vielbein. The solution has to be constructed, it cannot be given a priori. The choice of spherical symmetry is related to he fact that any astrophysical body has spherical symmetry. The Friedmann solution, the one giving the expansion of the Universe, is an expanding solution. In the Kerr solution we have a cylindrical symmetry. We must require a particular solution before approaching to solve the equations.

The Schwarzschild solution is not only a prototype of spherical symmetry but it is also the prototype of a black hole. The orbits can be derived taking into the account the metric and constructing the Lagrangian related to the metric and solving the Euler-Lagrange equations. We get then that orbits are related to conservation of energy and angular momentum. They are in agreement with Kepler and Newton laws.
The Schwarzschild solution can be recast in polar coordinates as well.

VII. Lesson 7

I. The Schwarzschild solution

Schwarzschild solution means we are assuming a symmetry for the Einstein field equations because we are not capable of solving in general this kind of equations. The general approach is to choose a priori the symmetry, we impose it on the field equations. Symmetries are much more general then solutions and this is why we can impose a symmetry on a class of solutions.
Spherical symmetry means that spacetime around a given object is spherically symmetric. If the object is rotating we have to choose cylindrical symmetry. The trick is to find classes of solutions according to what we need. This can be done when we know a priori the symmetry. If we have two interacting stars, the symmetry is not so clear, and we need numerical methods.
The number of solutions of Einstein solutions is not so large because we do not know the symmetry in most cases.

Let us construct the spherical symmetry. It gives rise to the possibility to obtain a black hole. Moreover, there is an important feature related with spherical symmetry which is the Birchoff theorem (the solution is stationary and static). We cannot have emission of GWs from a body in spherical symmetry. For GWs we need at least a quadrupolar structure which is not given in spherical symmetry.
Spherical symmetry means the metric has to be spherically symmetric: the entries of the metric do not depend on angles, but only on time and space. Therefore, the most general metric related with spherical symmetry is given by:

$$ds^2 = A(r,t)c^2dt^2 - B(r,t)dr^2 - 2C(r,t)drdt - D(r,t)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{31}$$

For the sake of generality we also considered off-diagonal terms, but in any case we can apply a specific transformation to discard these terms (the Jordan rule for matrices, which is the diagonalization). Here the diagonalization is quite easy because we do not have dependence on angles, otherwise it would not be possible. SO we can recast the line element in the form:

$$ds^2 = e^{\nu}c^2dt^2 - e^{\lambda}dr^2 - r^2d\Omega^2 \tag{32}$$

This form was supposed by Schwarzschild because we need a priori to avoid singularities. Moreover if the exponents are zero we can recover the Minkowski metric.
We need the Ricci tensor. We want to construct the external solution: we need the Ricci tensor outside the spherical body. So we need to calculate the Christoffel symbols. To do this we can take start from the Lagrangian constructed from the metric. From the Euler-Lagrange equations we can recover the geodesic equation and from it we can state which Christoffel symbols are different from zero. Once we have the Ricci tensor we can construct the field equations. A much more quick approach is based on the vielbein. Nevertheless, this is the canonical approach.

There is an interesting story about how the solution was found out. Schwarzschild was a mathematician, he was a soldier during WWI on the Russian front (1916). He sent his calculations to Einstein in Berlin, and he died during the battle of Tunnenberg. He couldn't see his results published.
The interesting components are R_{00}, R_{11} and R_{01} because we do not have dependence on angles. λ is the coefficient of the radial part of the metric and it has to be independent of time. Einstein said *Tensor calculus is killing General Relativity.*

We find an interesting result: the metric does not depend on time. It is in agreement with the Birchoff theorem. **The gravitational field related with spherical symmetry is not evolving**. This is related to the fact that we cannot have emission of any radiation.
We also have to recover asymptotic flatness, i.e. Minkowski spacetime. Therefore, $e^{\lambda} \to 1$ and $e^{\nu} \to 1$ when $r \to \infty$. Moreover, the fact that it should be

$$e^{\nu} = e^{-\lambda} \tag{33}$$

has a very important physical meaning.
The potential has to be negative defined in order to recover the Newton potential:

$$e^{\nu} = g_{00} = 1 - \frac{2U}{c^2} = 1 - \frac{2GM}{rc^2} = 1 - \frac{R_S}{r} \tag{34}$$

The final form of the metric is:

$$ds^2 = \left(1 - \frac{R_S}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{R_S}{r}\right)} - r^2 d\Omega^2 \tag{35}$$

In the weak field limit we recover exactly what we get when considering the geodesic evolution. Moreover, when the physical radius approaches the Schwarzschild radius we get a singularity. We get $1-1$, so we cannot define time and we are losing the size of the object. When we are in a black hole, we are losing the definition of time and the size of objects: in a BH we don't have a notion of time and any object becomes point-like. We get an intrinsic singularity which is a BH. This is a general trend: when a system reaches its gravitational radius it becomes a BH. Any object with a mass has a gravitational radius. But when we approach it physics is changing. The Schwarzschild solution is not a peculiarity but the prototype of any kind of BH. However, for physical BHs we have to take into account rotation and/or charge.
For a BH we need mass, angular momentum and radius for determining its size. When we have angular momentum the horizon is split in two.
BHs are exact solutions of Einstein field equations and they are physical systems! A BH is a gravitational field, a standard compact object. Up to now people considered BHs just as particular objects related with mathematical results. It is not so, because a BH is a standard object in the sky which physical properties are related to its radius.

Take the picture of the BH (Messier87). We see the event horizon where we lose information, because light is so deviated that we lose the possibility to get information from the inside. A BH is so compact that light cannot come out, it's a solution to Einstein field equations. A singularity is because we lose the standard physics, but it is mathematically consistent.

The discovery of GWs is related to the fat that two interacting BHs get rise to the emission of GWs, because we are no longer in spherical symmetry. The Schwarzschild solution is just a prototype to a standard object.
A NS is different from a BH because the radius of a NS is greater than its Schwarzschild radius. In a NS we have the last stage of stable matter: we stabilize the object with Fermi pressure (we are occupying all the states we have according to Pauli principle, since neutrons are fermions). If the mass is larger we cannot provide enough pressure: larger masses collapse. The last stable system is of order of the Chandrasekhar mass ($1.44 M_\odot$) which gives the last stable object before becoming a BH.

How can we determine a singularity? **What is the signature of a singularity?** The observational signature is absence of direct light emission. Mathematically we have a divergence. The physical signature is the event horizon: there is an horizon. We can have a singularity in the case in which an horizon exists. We can also have singularities without horizons, which are fully consistent with Einstein's field equations. A worm hole is a singularity without an event horizon, because we can pass through a worm hole and recover the standard physics on the other side of the singularity.
We can classify singularities:

- **Bare singularities**: objects without an event horizon
- **Standard singularities** where we have an horizon

When we have a singularity we have no signal at a certain point and this is the horizon where we lose the causal structure of spacetime.
On the other hand, we are capable of passing through a singularity, because it is allowed by the Einstein field

equations (Einstein-Rosen solution for example). Dirac said if we have solutions to a beautiful equation this solution must exist. *If a theory is self-consistent, everything inside the theory must exist.*

Black holes are physical objects! First of all because they are exact solutions of Einstein field equations. Second they are consistent with physics and they have been observed!

Naked singularities: in order to be a physical singularity there must be some requirements. We need traversability and stability. We need to understand what kind of energy conditions we have to take care of. The energy conditions tell us when the stress-energy tensor is physical: it has to be a perfect fluid. We cannot achieve this kind of singularity in a standard perfect fluid. By standard matter we cannot achieve a worm hole, forbidden by energy conditions. If we have non standard matter, like with no geometrical dofs or some kind of dark matter, we can write the energy conditions so that we can get the worm hole satisfying traversability and stability.
In order to traverse a worm hole some consistency conditions must be satisfied and these conditions are related with energy.

A tesseract is another type of singularity, a higher dimensional one.

So we need consistency with field equations, classification and **detectability**. For now we only detected BHs. A singularity is a solution of Einstein field equations where for certain range of parameters we get singular physics. Beside we need to understand the features of the singularity. For a naked singularity we cannot have an horizon. Singularity also means that we are changing topology. If we are inside the Schwarzschild radius time is no longer Lorentzian, but follows an Euclidean definition. We get an imaginary time. For an observer outside time is perceived differently. For us Euclidean time is nothing, but this is no such for the observer inside the singularity. One of the biggest issue is trying to classify singularities.

Coordinate singularity means that we can remove the singularity somehow. A true singularity is related to physical features.

VIII. Lesson 8

i. Experimental test of General Relativity

Experimental tests are important to show that GR is a physical theory. Recently the full success has been achieved with the GW discovery and the BH picture. It showed that geometrical description of spacetime can be achieved in the framework of GR. The main issue is the fact tat GWs represent the fact spacetime is a dynamical quantity: perturbations of spacetime are propagating. On the other hand, BHs can actually be real solutions, and not just singularities.
Let's review The test in chronological order.

i.1 Mercury perihelion precession

If we consider the system made of Sun and Mercury, its orbit is precessing. It can be understood if we consider all the possible perturbations: Venus, seismic eruptions on the Sun, etc. Starting from from classical considerations, it has come out that summing all perturbation w ehave an advance of $574''$ per century. This can be understood in the framework of classical theory up to $531''$.

The first to perform this calculation was Le Verrier. In Paris observatory all his notes are conserved. He addressed the $531''$. In the framework of Newtonian theory we cannot address $43''$. The precision of Newtonian theory is actually high! It was also thought that there was another planet between Mercury and Sun.
With GR, considering trajectories of the planet around the Sun, was capable to address exactly the $43''$ precession. This calculation was done by Einstein and Eddington. They considered a perturbation of the orbits, taking into account a Schwarzschild solution and we consider orbits around a Schwarzschild solution [*See notes*]. The correction is given exactly!
In general, the amount of precession can be a test of GR and of any theory of gravity.

i.2 The deflection of light

The main investigator (1919) was Eddington, who showed that during an eclipse of the Sun the deviation of light rays could be calculated according to GR.

Nowadays, gravitational lensing is an active area (Euclid mission). We can weigh matter according to the deviation. It is a good tool to estimate the amount of dark matter in a system. **The gravitational field behaves as an optically active medium**. So we can calculate the refraction and diffraction related to the gravitational field. When a light ray is moving in the gravitational field, it is deviated and the amount of deviation is analogue to change the density of the medium. We can apply the geometrical optics to this kind of situations.

Gravitational lensing is not gravitational astronomy, because in gravitational lensing we are using anyway the electromagnetic signals. It's an optical system. Gravitational astronomy means that we want to get information taking into account gravitational signals. Combining these information we get multi-messenger astronomy (EM signals, neutrinos and GWs).

i.3 The gravitational redshift

It was predicted by Einstein in 1907 before the formulation of the theory. It says that the spectral line redshift is induced by the gravitational field. The presence of a gravitational field gives rise to a shift, which is a redshift. **The Gravitational field is slowing the emission or absorption lines in a give system**. This is a consequence of the EP.
The spectral line redshift is induced by the geodesic motion of particles. It's a confirmation of the EP. It's a

displacement in spectral lines due to the gravitational interaction generated by a object.

Pound and Rebka performed the test and it was **the first precision experiment** of GR. They put some atoms of iron on the Jefferson tower (Harward) and they let them fall down. These atoms underwent a sort of Mossbauer effect which amount was related with the height of the tower and so with the terrestrial gravitational field. The change in the Mossbauer effect was related to g.

i.4 Relativistic Cosmology

We have the exact date when relativistic Cosmology was born: October 1917. It was based on the lecture *Cosmological considerations on the General Relativistic theory*, given by Einstein, director of the Keiser Wilhelm Institute at the time (Berlin).

Before Einstein, Cosmology was seen as a branch of Philosophy. There is also the so called Einstein universe: static system with a cosmological constant stopping the evolution of the system. The important fact is the method: for the first time Cosmology was considered as a part of science and not just related to metaphysics. Now we are saying that a set of equations describing spacetime is describing the shape of the Universe.
In 1929 Hubble discovered the expansion of the Universe: nebulae are not in our Galaxy. Soon after this, Friedmann was capable to demonstrate that the Universe is in expansion. This is the start of observational Cosmology. It can be formulated under the standard of a mathematical framework and there are also observations.

Lemaitre, Friedmann, Robertson and Walker formulate the Big Bang theory: if we evolve Einstein equations we get a singularity: the original black hole, the Big Bang. This was confirmed by some observations: Big Bang nucleosynthesis. The main observational discovery is related to the CMB, discovered by Penzias and Wilson. Peebles realized that this radiation was the remnant of the initial Big Bang, and he was able to calculate exactly the amount and temperature of this radius assuming the Universe as a huge black body. After this discovery the claim was that **the Universe is a thermodynamic system**.

We also had the discovery of the large scale structure: the disposition of galaxies and clusters of galaxies. This disposition is related to the primordial cosmic perturbations. The amount of anisotropy in the initial temperature is of order 10^{-5}. If we compare the density of a galaxy with the density of the background we get the same number! It is 10^{-24} for a galaxy, and $10^{-29} - 10^{-30}$ for the background. So we can rule out the amount of initial perturbation wrt the large scale structure.

The recent discovery, by Reese, Perlmutter and Schmidt, is **dark energy**. The Universe is expanding and the expansion is not decelerating as forecast by GR, but it is an accelerated expansion. There is a sort of dark energy giving rise to acceleration.
The main issue is the problem of dark energy and dark matter. There are several probes of GR, which are self-consistent.

i.5 Picture of a black hole

It's confirming the fact that we have the probe that light is falling down inside the BH and dynamics around BH is governed by GR. It's one of the main probes: BHs are real!

i.6 Gravitational waves

They are the main confirmation of GR, forecast by Einstein with two papers in 1916 and 1918. He didn't believed in them. Due to the fact that we have a high suppression term, it is very very difficult to detect them.
A GW is a small perturbation of spacetime that like an EM wave propagates at light speed. For EM a vector

propagates, while here it is a tensor to propagate. They travel in space without absorption or ri-emission. It's a formidable probe to test spacetime at any epoch and length.

A GW is characterized by two polarizations, quadrupolar nature, they are generated by tidal effects of astrophysical objects, they are related with very massive objects. Coalescing binaries of BHs was related to the first detection.

Indirectly, Hulse and Taylor detected an interesting behavior of a pulsar PSR1913+16. The emission of GWs reduced the energy of the orbital motion: the two stars approach each other and in doing so they lose some signal. Subtracting the EM signal we get the gravitational signal. They were not able to give the structure of GWs, the wavenumber or amplitude.