High Energy Astrophysics
-Theory-

Pasquale Blasi, GSSI
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written by Ulyana Dupletsa

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i. Introduction

Most of the Universe, our Galaxy, the space in between galaxies, environment of things like supernova explosion or gas around the neutron star or a black hole, they are in the Universe in the regime of a plasma.
Plasma means, at zero order, that all the particles in the system are free charges. **Plasma is a gas of electrons and protons in charge equilibrium**. We can imagine to have a bunch of atoms, with electrons and protons, and suddenly increase the temperature, until the level at which electrons detach from the atoms and start moving as free charges. For the hydrogen, for example, this happens at temperatures $T > 10^4$K. Most of the interstellar medium has a temperature of order $10^4 - 10^5$K with a density of 1 particle per cm^3. In astrophysical contexts, densities are always extremely small, which is the characteristic that makes plasma a really good metal, i.e. it is very conductive. As we will see soon, we cannot create electric fields in a plasma, as having an electric field would set electrons and protons in motion, and the electric field is short circuited.

We are now to construct a simple model to describe what happens to a gas of electrons and protons and if there are special properties. What will these charges do? They will generate electric and magnetic fields and interact of consequence. In a generic plasma the electric and magnetic fields are produced by the same particles that we are trying to describe. On top of this, there might be some large scale field. For instance, the plasma of the interstellar medium of our galaxy feels the magnetic field of our galaxy (the large scale magnetic field).
We want to start with a toy model, the simplest case of a plasma made of protons and electrons (a more complicated situation would include, for example, also helium). In this system there will a magnetic and an electric filed which satisfy Maxwell's equations:

REMINDER: Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = 4\pi\zeta, \quad \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

Gaussian Units

Pay attention that throughout these notes we are always using the Gaussian units. With respect to SI units, the Gaussian units define the unit of charge is defined using a dimensional combination of the mechanical units (gram, centimeter, second) as:

$$1\text{statC} = 1\text{Fr} = 1\text{cm}^{3/2}\text{g}^{1/2}\text{s}^{-1}$$

where **statC** stands for *statcoulomb* and **Fr** stand for *Franklin*.
Therefore, given the unit of charge in in SI units as 1.602×10^{-19}C, in gaussian units it becomes:

$$e = 4.8 \times 10^{-10}\text{cm}^{3/2}\text{g}^{1/2}\text{s}^{-1}$$

Similarly, the magnetic field in gaussian units is measured in gauss:

$$1\text{gauss} = 1\text{cm}^{-1/2}\text{g}^{1/2}\text{s}^{-1}$$

The source terms are ζ (charge density of the system) and $\vec{J}$ (current density). We shall describe the protons and electrons through the charge density and the current that they carry and the electric and magnetic fields are the ones that they produce themselves

$$i \rightarrow \text{ions (i.e. protons)}$$
$$e \rightarrow \text{electrons}$$

Then the mass density of the material, charge and current densities, can be written as

$$\rho = m_i n_i + m_e n_e \tag{1}$$

$$\zeta = n_i e - n_e e = (n_i - n_e)e \tag{2}$$

$$\vec{J} = n_i e \vec{v}_i - n_e e \vec{v}_e \tag{3}$$

If we leave the system by itself, it will acquire some density which will be common to the density of electrons and protons (there is no reason why they should be different). Let's call n_0 the density of ions and electrons at equilibrium

$$n_0 = n_i = n_e$$

i.1 Plasma screening effect

What happens if we perturb a little charge equilibrium, by adding a single charge? Let's say that is a positive charge. The charge density becomes

$$\zeta = (n_i - n_e)e + e\delta(\vec{r}) \tag{4}$$

From Maxwell's equations we have that the electric field can be written from the potential $\vec{E} = -\vec{\nabla}\phi$, therefore the first Maxwell's equation becomes

$$-\vec{\nabla}^2\phi = 4\pi(n_i - n_e)e + 4\pi e\delta(\vec{r}) \tag{5}$$

We find a second order differential equation for the potential and the delta function can be used just as a boundary condition. So we will solve the equation for all $\vec{r} \neq 0$ and then we will use the boundary condition for $\vec{r} = 0$. If the system was in equilibrium before the charge was added, the distribution of electrons and protons after the addition of one charge will be only slightly perturbed. Therefore we can write the density of electrons and ions after the addition of the single charge using the Boltzmann distribution:

$$n_e = n_0 \exp\left[-(-e\phi)/kT\right] \tag{6}$$

and

$$n_i = n_0 \exp\left[-e\phi/kT\right] \tag{7}$$

Note that ϕ is a function of $\vec{r}$, the distance from the source.
We substitute into equation (5) and try to find an expression for ϕ, under the hypothesis that the perturbation is small ($e\phi \ll kT$, the electric energy subjected to particles because of the addition of a single particle is much smaller than typical thermal energy of the particle), so that we can Taylor expand at first order the exponentials:

$$\begin{aligned}
-\vec{\nabla}^2\phi &= 4\pi e n_0 \exp\left[-e\phi/kT\right] - 4\pi e n_0 \exp\left[e\phi/kT\right] + 4\pi e\delta(\vec{r}) \\
-\vec{\nabla}^2\phi &= 4\pi n_0 e\left[\cancel{1} - \frac{e\phi}{kT}\right] - 4\pi e n_0\left[\cancel{1} + \frac{e\phi}{kT}\right] + 4\pi e\delta(\vec{r}) \\
-\vec{\nabla}^2\phi &= -8\pi n_0 \frac{e^2\phi}{kT} + 4\pi e\delta(\vec{r})
\end{aligned} \tag{8}$$

If we are not in $\vec{r} = \vec{0}$, we have that the potential must satisfy the following equation

$$\vec{\nabla}^2\phi = 8\pi n_0 \frac{e^2\phi}{kT} \tag{9}$$

We can easily solve it. We write the Laplacian in spherical coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\phi}{\partial r}\right]=\frac{8\pi n_0 e^2}{kT}\phi \tag{10}$$

and use the identity[1]

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\phi}{\partial r}\right]=\frac{1}{r}\frac{\partial^2}{\partial r^2}\left[r\phi\right] \tag{11}$$

$$\begin{aligned}\frac{1}{r}\frac{\partial^2}{\partial r^2}\left[r\phi\right]&=\frac{8\pi n_0 e^2}{kT}\phi\\ \frac{\partial^2}{\partial r^2}\left[r\phi\right]&=\frac{8\pi n_0 e^2}{kT}r\phi\end{aligned} \tag{12}$$

Calling $r\phi = f$

$$f=f_0\exp\left(-\frac{r}{\lambda}\right) \tag{13}$$

with

$$\lambda^2=\frac{kT}{8\pi n_0 e^2} \tag{14}$$

So

$$\phi=\frac{f}{r}=\frac{f_0}{r}\exp\left(-\frac{r}{\lambda}\right) \tag{15}$$

To find the constant we use the boundary condition, which says that if we are sufficiently far away from the place where we put the charge the potential has to go to zero. On the other hand, if we go at infinitesimal distance from the source we have to recover the Coulomb potential, which is e/r. Therefore

$$\phi=\frac{e}{r}\exp\left(-\frac{r}{\lambda}\right) \tag{16}$$

What have we obtained? By perturbing our system we create an electric potential that is felt by the other particles only if they find themselves within a distance λ from the source we have added. If $r \gg \lambda$ we can just ignore this potential. **That means electric fields in a plasma can be created only on short scales.** On larger scales there is a sort of automatic **screening effect**.
We shall define the **Debye length** as follows

$$\lambda_{\mathrm{D}}=\left[\frac{kT}{4\pi n_0 e^2}\right]^{1/2}=\sqrt{2}\lambda \tag{17}$$

The Debye length is a fundamental scale in plasmas that tells us that if we perturb a plasma, the perturbation can only be felt through the electric fields on very short distances from the added charge. We can say that we can create

[1]

$$\begin{aligned}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\phi}{\partial r}\right]&=\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial r^2}\\ &=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}\\ &=\frac{1}{r}\frac{\partial}{\partial r}\left(\phi+r\frac{\partial\phi}{\partial r}\right)\\ &=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial r}{\partial r}\phi+r\frac{\partial\phi}{\partial r}\right)\\ &=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\left(r\phi\right)\right)\\ &=\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\phi\right)\end{aligned}$$

electric fields on the Debye length, which is a very short distance for plasmas in astrophysics.

Let us see how small it is compared to the number of particles we can have at these scales. We want to collect some statistical properties of a plasma system and **to do statistics we need a lot of particles**. **What is the number of particles inside a Debye length?** To this scope we define the following **plasma parameter** Λ_e and we require it to be large enough

$$\Lambda_e \simeq n_0 \lambda_{\mathrm{D}}^3 = n_0 \left[\frac{kT}{4\pi n_0 e^2} \right]^{3/2} = \left(\frac{kT}{4\pi e^2} \right)^{3/2} n_0^{-1/2} \gg 1 \tag{18}$$

This condition is easy to fulfill if the densities are low, which is generally always the case for plasmas.

i.2 Hypothesis of collisionless plasma

Here is another question we can ask: generally particles in a plasma interact through electric and magnetic fields that they create. Therefore such a system will evolve quicker because particles are suffering very close encounters due to Coulomb interactions or rather because the particles are just feeling large scale fluctuations in the electric and magnetic fields? We answer to this question by comparing two scales of energy transfer:

- one is how energy is transferred through a **Coulomb scattering**: we have two particles approaching each other with some impact parameter b, the kinetic energy changes by an amount in modulus equal to

  $$\frac{e^2}{b}$$

  This quantity will be small or bug depending on how it compares with thermal energy
- **thermal energy** given by

  $$3kT$$

Let's compare the Coulomb energy (e^2/b) with the typical thermal energy ($3kT$). We have that the two energy transfer process are comparable if the b parameter is of order

$$b = \frac{e^2}{3kT} \tag{19}$$

To understand if this is common or not, let us compare b with the average distance between two particles in a plasma (given the density n_0, the average distance will be $n_0^{-1/3}$)

$$\frac{b}{n_0^{-1/3}} = \frac{e^2}{3kT} n_0^{1/3} = \frac{4\pi e^2}{kT} \frac{1}{12\pi} n_0^{1/3} \sim \Lambda_e^{-2/3} \ll 1 \tag{20}$$

The only way to have a lot of energy exchange between two particles in a plasma is that they approach each other at a distance which is much smaller than the typical distance between two particles. So clearly this is not very common. **Therefore plasma really evolves on large scales not because of spikes in the electric field when two particles get very close, but rather because there are collective effects that create electric and magnetic fields, due to collective motion**.

We are describing here **collisionless plasmas**. This makes sense for astrophysical purposes. In the interstellar medium, in fact, the densities are about 1 particle per cm^3. The cross section for electromagnetic interaction can be assumed to be the Thompson scattering $\sigma \sim 10^{-24}\mathrm{cm}^2$. The distance between two interactions is of order $\frac{1}{n\sigma} \sim 10^{24}$, which is roughly the distance to Andromeda. So in the interstellar medium, the distance between two collisions, if the particles were to collide, would be gigantic and this does not make any sense. **So the rate of collisions is negligible.**

Summing up ...

- Plasma is a state of matter where electrons and protons move as free charges (at first order)
- A plasma system creates a screening effect for electric fields, which can be felt only on the Debye scales

$$\phi = \frac{e}{r} \exp\left(-\frac{r}{\lambda}\right)$$

with

$$\lambda_{\rm D} = \left(\frac{kT}{4\pi n_0 e^2}\right)^{1/2}$$

- A plasma is a collisionless system which evolves due to collective effects.

II. Plasma waves

We now want to derive a dispersion relation which can tell us which kind of waves are allowed to propagate in a plasma.
To this purpose we define now the **conductivity matrix**, $\hat{\sigma}$:

$$\vec{J} = \hat{\sigma}\vec{E}, \qquad (J_r = \sigma_{rs} E_s) \tag{21}$$

and introduce the **displacement vector**, $\vec{D}$, which is defined such that:

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \tag{22}$$

$$\vec{J} = \frac{1}{4\pi}\left[\frac{\partial \vec{D}}{\partial t} - \frac{\partial \vec{E}}{\partial t}\right] \tag{23}$$

The electric field can be written as a sum over modes:

$$\vec{E}(\vec{x}, t) = \frac{1}{(2\pi)^3}\int d^3k \tilde{E}(\vec{k}, \omega) \exp\left[-i\omega t + i\vec{k}\cdot\vec{x}\right] \tag{24}$$

Now we'll use a common trick and go in the Fourier space. As long as the system remains in its linear regime this is a very useful technique. We can now write (23) in Fourier space:

$$\tilde{J}_r = -\frac{i\omega}{4\pi}\left[\tilde{D}_r - \tilde{E}_r\right] \tag{25}$$

The definition (21) also holds in Fourier space

$$\begin{aligned} \sigma_{rs}\tilde{E}_s &= -\frac{i\omega}{4\pi}\left[\tilde{D}_r - \tilde{E}_r\right] \\ \tilde{D}_r &= \frac{4\pi}{i\omega}\left[\frac{i\omega}{4\pi}\tilde{E}_r - \sigma_{rs}\tilde{E}_s\right] = \tilde{E}_r + i\frac{4\pi}{\omega}\sigma_{rs} \end{aligned} \tag{26}$$

Writing $\tilde{E}_r = \delta_{rs}\tilde{E}_s$ we end up with

$$\tilde{D}_r = \left[\delta_{rs} + i\frac{4\pi}{\omega}\sigma_{rs}\right]\tilde{E}_s \equiv \hat{\mathbb{K}}_{rs}\tilde{E}_s \tag{27}$$

where we define $\hat{\mathbb{K}}$ the **dielectric tensor**.
Now let us take the curl of the Maxwell's equation with the curl of $\vec{E}$

$$\begin{aligned}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) &= \vec{\nabla}\times\left(-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\right)\\ \vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right)\\ &= -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial\vec{D}}{\partial t}\right)\\ &\overset{F.T.}{=} \frac{\omega^2}{c^2}\frac{1}{(2\pi)^3}\int d^3k\tilde{D}(\vec{k},t)\exp(\dots)\end{aligned} \tag{28}$$

In this way we find the **dispersion relation** to be

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \frac{\omega^2}{c^2}\hat{\mathbb{K}}\cdot\vec{E} \tag{29}$$

When we are going to Fourier space, in fact, we are interpreting each quantity as an overlap of waves with a given frequency. In general, not all of these frequencies are allowed in a system, but only some are allowed and the ones that are allowed are the solutions of what we call the dispersion equation. **Therefore we have just found the dispersion relation for the perturbations that are allowed in our system**.
Since protons are much heavier than electrons, let us consider the **electron current**. The protons are stuck and the only currents are those induced in electrons. That means that for one electron of mass m_e and velocity $\vec{v}$, the equation of motion (newtonian approximation!) is given by

$$m_e\frac{d\vec{v}}{dt} = -e\vec{E} \xrightarrow[F.T.]{} -i\omega m_e\tilde{v}_r = -e\tilde{E}_r \tag{30}$$

The electrons are able to move and the current is then given by

$$\vec{J} = -n_e e\vec{v} \overset{F.T.}{=} -n_e e\tilde{v}_r = -\frac{n_e e^2}{i\omega m_e}\tilde{E}_r \tag{31}$$

We have just found the expression for the conductivity:

$$\tilde{J}_r = \frac{i n_e e^2}{\omega m_e}\tilde{E}_r = \hat{\sigma}_{rs}\tilde{E}_s \tag{32}$$

That is Omh's law. Then we have just found the conductivity σ in terms of physical parameters of our plasma:

$$\hat{\sigma}_{rs} = \frac{i n_e e^2}{\omega m_e}\delta_{rs} \tag{33}$$

and

$$\mathbb{K}_{rs} = \delta_{rs} + \frac{4\pi i}{\omega}\sigma_{rs} = \delta_{rs}\left[\mathbb{1} - \frac{4\pi e^2 n_e}{m_e\omega^2}\right] = \delta_{rs}\left[\mathbb{1} - \left(\frac{\omega_p}{\omega}\right)^2\right] \tag{34}$$

with

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} \tag{35}$$

which is called **plasma frequency**.
Now we can rewrite the dispersion relation as follows

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \frac{\omega^2}{c^2}\left[\frac{\omega^2-\omega_p^2}{\omega^2}\right]\vec{E} \quad \xrightarrow[F.T.]{} \quad \vec{k}\times\left(\vec{k}\times\tilde{E}\right) + \frac{\omega^2-\omega_p^2}{c^2}\tilde{E} = 0 \tag{36}$$

Note that the i^2 coming from Fourier transforming cancels out with the minus sign coming from bringing to the LHS the ω terms.
This is the dispersion relation we needed. It is difficult to solve it in its generality, but we are going to split the solutions in two bunches.
We are to classify the modes distinguishing between those parallel to the electric field ($\vec{k} \parallel \vec{E}$) and those perpendicular to it ($\vec{k} \perp \vec{E}$)

A generic electric field can be written as

$$\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp}$$

$$\vec{k} \times \vec{E}_{\parallel} = 0 \quad \text{and} \quad \vec{k} \cdot \vec{E}_{\perp} = 0$$

Therefore we obtain[2]

$$\begin{aligned} \left[\omega^2 - \omega_p^2\right] \vec{E}_{\parallel} = 0 \\ \left[-c^2k^2 + (\omega^2 - \omega_p^2)\right] \vec{E}_{\perp} = 0 \end{aligned} \tag{37}$$

We observe that the only longitudinal modes that can propagate are the ones that **must have the plasma frequency**.
On the other hand, the only transverse modes that can propagate are those satisfying the condition

$$\omega^2 = \omega_p^2 + c^2k^2 \implies k^2 = \frac{1}{c^2}\left(\omega^2 - \omega_p^2\right) \tag{38}$$

If $\omega < \omega_p$, then $k^2 < 0$, and we have two purely imaginary solutions (big mess)

$$k = \pm\frac{i}{c}\left(\omega_p^2 - \omega^2\right)^{1/2} \sim \pm\frac{i}{c}\omega_p$$

One is explosive and therefore not physical, whereas the other one is exponentially suppressed over a spatial scale called skin depth. The perturbation dies very quickly. We define the **skin depth** of the plasma the quantity

$$\Delta x = \frac{c}{\omega_p} \tag{39}$$

We can find the relation between the skin depth Δx and the Debye length λ_D. First we can rewrite the plasma frequency as

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} = \left(\frac{4\pi n_e e^2}{kT}\frac{kT}{m_e}\right)^{1/2} = \frac{1}{\lambda_\text{D}}\left(\frac{kT}{m_e}\right)^{1/2} \tag{40}$$

It easily follows that

$$\Delta x = \frac{c}{\omega_p} = c\lambda_\text{D}\left(\frac{kT}{m_e}\right)^{1/2} = \lambda_\text{D}\sqrt{\frac{m_e c^2}{kT}} \tag{41}$$

Therefore

$$\frac{\Delta x}{\lambda_\text{D}} = \sqrt{\frac{m_e c^2}{kT}} \tag{42}$$

[2]Remember the following property of triple vector products:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \cdot \vec{c}) - \vec{c} \cdot (\vec{a} \cdot \vec{b})$$

Summing up ...

- The dispersion relation which classifies the modes allowed to propagate in a plasma is given by

$$\vec{k} \times \left(\vec{k} \times \tilde{E}\right) + \frac{\omega^2 - \omega_p^2}{\omega^2} \tilde{E} = 0$$

- Longitudinal and transverse modes are classified according to

$$\left[\omega^2 - \omega_p^2\right] \vec{E}_\parallel = 0$$
$$\left[-c^2 k^2 + (\omega^2 - \omega_p^2)\right] \tilde{E}_\perp = 0$$

  – Longitudinal modes only propagate if their frequency equals the plasma frequency

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2}$$

  – Transverse modes can propagate if they satisfy

$$k^2 = \frac{1}{c^2}\left(\omega^2 - \omega_p^2\right)$$

  Transverse perturbations with $\omega < \omega_p$ die quickly and are characterized by a skin depth given by

$$\Delta x = \frac{c}{\omega_p} = \lambda_\mathrm{D} \sqrt{\frac{m_e c^2}{kT}}$$

III. Vlasov description of collisionless plasma

We start with just one particle (remember that usually a plasma at equilibrium is neutral) and describe its motion in phase space, where the particle is identified through the coordinates $(\vec{x}, \vec{v}, t)$. Moreover, we assume a **non relativistic regime**. Then the number of particles can be written as

$$N_s\left(\vec{x}, \vec{v}, t\right) = \sum_{i=0}^{N_0} \delta\left(\vec{x} - \vec{X}_i(t)\right) \delta\left(\vec{v} - \vec{V}_i(t)\right) \tag{43}$$

where the pedex s identifies the species, ions and electrons: $s = i, e$.
The objective is to describe the motion of ions and electrons under the action of electric and magnetic fields that the particles produce themselves. Note that in this system to describe the motion of a particle we need to know what all the other particles are doing. Moreover, the particle we are describing do not interact through collisions.
Let's start from (43) and differentiate with respect to time

$$\frac{\partial N_s}{\partial t} = -\sum_{i=1}^{N_0} \dot{\vec{X}}_i \cdot \vec{\nabla}_x \left(\delta\left(\vec{x} - \vec{X}_i(t)\right) \delta\left(\vec{v} - \vec{V}_i(t)\right)\right) - \sum_{i=1}^{N_0} \dot{\vec{V}}_i \cdot \vec{\nabla}_v \left(\delta\left(\vec{x} - \vec{X}_i(t)\right) \delta\left(\vec{v} - \vec{V}_i(t)\right)\right) \tag{44}$$

Note that all these deltas are 3-dimensional[3]
$\dot{\vec{v}}_i$ is the acceleration. Since the system is not relativistic, it can be related to a force, which is the **Lorentz force**[4]

$$m_s \dot{\vec{V}}_i = \left[q_s \vec{E}^M(\vec{x}, t) + q_s \frac{\vec{V}_i(t)}{c} \times \vec{B}^M(\vec{x}, t) \right] \Bigg|_{\vec{x}=\vec{X}_i} \tag{45}$$

The apex M here stands for *microscopic*, for $\vec{E}^M$ and $\vec{B}^M$ are the **microscopic electric and magnetic fields produced by the particles**. So we need equations that describe the electric and magnetic fields each particle is feeling because of all the other particles. These are the Maxswell's equations:

$$\vec{\nabla} \cdot \vec{E}^M = 4\pi\zeta^M(\vec{x}, t), \qquad \vec{\nabla} \cdot \vec{B}^M = 0 \tag{46}$$

$$\vec{\nabla} \times \vec{E}^M = -\frac{1}{c}\frac{\partial \vec{B}^M}{\partial t}, \qquad \vec{\nabla} \times \vec{B}^M = \frac{1}{c}\frac{\partial \vec{E}^M}{\partial t} + \frac{4\pi}{c}\vec{J}^M \tag{47}$$

The source terms (current and charge density) are given by

$$\zeta^M(\vec{x}, t) = \sum_{s=i,e} q_s \int d^3\vec{v} N_s(\vec{x}, \vec{v}, t) \tag{48}$$

$$\vec{J}^M(\vec{x}, t) = \sum_{s=i,e} q_s \int d^3\vec{v} N_s(\vec{x}, \vec{v}, t)\vec{v} \tag{49}$$

We shall now rewrite (44) using the expression for the acceleration

$$\begin{aligned} \frac{\partial N_s}{\partial t} = &- \vec{v} \cdot \vec{\nabla}_x \sum_{i=1}^{N_0} \delta\left(\vec{x} - \vec{X}_i(t)\right) \delta\left(\vec{v} - \vec{V}_i(t)\right) - \\ &- \sum_{i=1}^{N_0} \frac{q_s}{m_s} \left[\vec{E}^M + \frac{1}{c}\vec{V}_i \times \vec{B}^M \right] \cdot \vec{\nabla}_v \delta\left(\vec{x} - \vec{X}_i(t)\right) \delta\left(\vec{v} - \vec{V}_i(t)\right) \end{aligned} \tag{50}$$

The quantities in the square parenthesis are assumed to be calculated at $\vec{x} = \vec{X}_i$ at time t.
We can write

$$\frac{\partial N_s}{\partial t}(\vec{x}, \vec{v}, t) + \vec{v} \cdot \vec{\nabla}_x N_s(\vec{x}, \vec{v}, t) + \frac{q_s}{m_s}\left[\vec{E}^M + \frac{1}{c}\vec{v} \times \vec{B}^M \right] \cdot \vec{\nabla}_v N_s(\vec{x}, \vec{v}, t) = 0 \tag{51}$$

If we introduce the total derivative in phase space, $(\vec{x}, \vec{v}, t)$, this is nothing else than

$$\frac{dN_s}{dt} = 0 \tag{52}$$

[3]Remember the following property of the δ-function:

$$\frac{d}{dx}\delta(x - x_0) = -\delta(x - x_0)\frac{d}{dx}$$

This can be easily proven using the test function $g(x)$:

$$\int_{\mathbb{R}} dx \frac{d}{dx}\delta(x - x_0)g(x) = \underbrace{\delta(x - x_0)g(x)}_{0} - \int_{\mathbb{R}} dx \delta(x - x_0)\frac{dg}{dx} = -\frac{dg}{dx}\bigg|_{x=x_0}$$

where the first term goes to zero since a test function has a compact support.
In general if T is a distribution and ϕ a test function:

$$\partial^\alpha T(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi)$$

[4]In general, the acceleration is due to the force the particles feel. Were we dealing with dark matter, for example, that would be the gravitational force. Since plasma experiences electromagnetic interaction, we have the Lorentz force.

that is an expression of the **Liouville theorem** (conservation of the number of particles in phase space). Equation (51) is called the **Klimontovich-Dupree equation**.
Note that $\vec{E}^M$ and $\vec{B}^M$ are characterized by wild fluctuations, which are leveled at scales larger than the Debye scale. Furthermore, we are not even interested in the motion of single particles, we want to describe statistical properties integrating over little volumes which are though large enough to retain some statistics. It is customary then to take ensemble averages[5] on scales l such that

$$n_0^{1/3} \ll l < \lambda_{\rm D} \tag{53}$$

We shall write

$$N_s(\vec{x},\vec{v},t) = f_s(\vec{x},\vec{v},t) + \delta N_s(\vec{x},\vec{v},t) \tag{54}$$

where f_s is the ensemble average. We want to adopt a perturbative approach here. So we write

$$\vec{E}^M(\vec{x},\vec{v},t) = \vec{E}(\vec{x},\vec{v},t) + \delta\vec{E}(\vec{x},\vec{v},t) \tag{55}$$

and

$$\vec{B}^M(\vec{x},\vec{v},t) = \vec{B}(\vec{x},\vec{v},t) + \delta\vec{B}(\vec{x},\vec{v},t) \tag{56}$$

On intermediate and large scales we can have **collective effects**. Moreover, if the system is in an external magnetic field (e.g. our own galaxy), the ensemble average will be felt, whereas fluctuations will average to zero.
So we have three quantities: $N_s, \vec{E}^M, \vec{B}^M$. We are going to substitute them in the Klimontovich-Dupree equation, bearing in mind that first order fluctuations averages to zero, but products of fluctuations (second order) do not:

$$\frac{\partial f_s}{\partial t}(\vec{x},\vec{v},t) + \vec{v}\cdot\vec{\nabla}_x f_s + \frac{q_s}{m_s}\left[\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right]\cdot\vec{\nabla}_v f_s = -\frac{q_s}{m_s}\langle\left(\delta\vec{E} + \frac{1}{c}\vec{v}\times\delta\vec{B}\right)\cdot\vec{\nabla}_v \delta N_s\rangle \tag{57}$$

The expression (57) is known as the **Kinetic plasma equation**. Here the electric and magnetic fields are only the ones on large scales.
We observe that $f_s, \vec{E}, \vec{B}$ are all first order approximations, whereas the RHS of (57) is second order and represents a **collision operator**. It describes what we can call collisionless collisions, what happens to a particle on small scales due to the effect of particles nearby. Therefore, at first order, the RHS is negligible and we have what is called the **Vlasov equation**:

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla}_x f_s + \frac{q_s}{m_s}\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)\cdot\vec{\nabla}_v f_s = 0 \tag{58}$$

The Vlasov equation[6] holds for a plasma system at equilibrium, i.e. for a collisionless system under the magnetic fields produced by the particles themselves.
Note that we have two Vlasov equations, one for each species (electrons and ions), but these are not two independent Vlasov equations, because the electric and magnetic fields, $\vec{E}$ and $\vec{B}$, that appear in each equation are due to all the particles in the system. That is to say that they are coupled through the electromagnetic fields produced by all the particles.
In addition to this, we can use Vlasov equation also for cosmic rays as they two are non collisional. Therefore, they have to be added as well to sources creating electric and magnetic field.
We can use the Vlasov equation to describe any system, provided that we know how to connect the source terms with the electromagnetic fields.
So we obtain a set of equations (let's consider the simpler situation with electrons and protons only) of the type:

$$\begin{cases} 1 \text{ equation for } s = i \\ 1 \text{ equation for } s = e \\ 4 \text{ Maxwell's eqs to connect } \vec{E} \text{ and } \vec{B} \text{ with the sources, } \zeta \text{ and } \vec{J} \end{cases} \tag{59}$$

[5] If we have a sufficiently large number of particles the ensemble average coincides with the average over the number of particles.
[6] If we have to apply the Vlasov equation to a relativistic system, all we have to do is to work with $(\vec{x},\vec{p},t)$ coordinates instead of $(\vec{x},\vec{v},t)$. In this way we do not have a mass m_s term.

This is a closed system of equation and can, in theory, be solved.

What about a relativistic Vlasov equation?

The relativistic Vlasov equation is written as:

$$\frac{\partial f_s}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f_s + \dot{\vec{p}} \cdot \vec{\nabla}_p f_s = 0$$

where

$$\dot{\vec{p}} = q_s \left[\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right]$$

The Maxswell's equations stay the same

$$\vec{\nabla} \cdot \vec{E} = 4\pi\zeta, \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J}$$

The source terms are, instead, given by

$$\zeta(\vec{x}, t) = \sum_{s=e,i} q_s \int d^3\vec{p} f_s(\vec{x}, \vec{p}, t)$$

$$\vec{J}(\vec{x}, t) = \sum_{s=e,i} q_s \int d^3\vec{p} f_s(\vec{x}, \vec{p}, t)\vec{v}$$

The only thing changing in the relativistic version is the $\dot{\vec{p}} \cdot \vec{\nabla}_p$ part and the integration in the source terms, which is now on the momentum space.

iii.1 Analysis of a particular case

Let us consider a system at equilibrium with ions and electrons only, described by the Vlasov equation:

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\alpha + q_\alpha \left[\vec{E} + \frac{1}{c} \vec{v} \times \vec{B} \right]_\beta \frac{\partial f_\alpha}{\partial p_\beta} = 0 \tag{60}$$

We put ourselves in the case where $\vec{E} = \vec{0}$, so that there is only a magnetic field on large scales (it can be the collective magnetic field of the interstellar medium)

$$\vec{B} = \vec{B}_0 \quad \text{large scale field} \tag{61}$$

We have an equilibrium state satisfying (60). Now we are to **impose a perturbation** and we want to find the dispersion relation of the type

$$F(k, \omega) = 0 \tag{62}$$

which will tell us what perturbations can propagate. In our system we have

$$\begin{cases} f_\alpha \to f_\alpha + \delta f_\alpha \\ E \to \delta E \\ B \to B_0 + \delta B \end{cases}$$

We adapt a perturbative approach at first order (neglect from second order on). Therefore:

$$\frac{\partial \delta f_\alpha}{\partial t} + \vec{v} \cdot \vec{\nabla} \delta f_\alpha + q_\alpha \delta \vec{E}_\beta \frac{\partial f_\alpha}{\partial p_\beta} + q_\alpha \frac{1}{c} \left[\vec{v} \times \delta \vec{B} \right]_\beta \frac{\partial f_\alpha}{\partial p_\beta} + q_\alpha \left[\frac{1}{c} \vec{v} \times \vec{B}_0 \right]_\beta \frac{\partial \delta f_\alpha}{\partial p_\beta} = 0 \tag{63}$$

From Maxwell's equations we have that

$$\vec{\nabla} \cdot \delta\vec{E} = 4\pi\zeta, \qquad \vec{\nabla} \cdot \delta\vec{B} = 0 \tag{64}$$

$$\vec{\nabla} \times \delta\vec{E} = -\frac{1}{c}\frac{\partial\delta\vec{B}}{\partial t}, \qquad \vec{\nabla} \times \delta\vec{B} = \frac{1}{c}\frac{\partial\delta\vec{E}}{\partial t} + \frac{4\pi}{c}\vec{J} \tag{65}$$

with

$$\zeta = \sum_{\alpha=e,i} q_\alpha \int d^3\vec{p}\delta f_\alpha \tag{66}$$

$$\vec{J} = \sum_{\alpha=e,i} q_\alpha \int d^3\vec{p}\delta f_\alpha \vec{v} \tag{67}$$

The usual trick is to move in the Fourier space, where

$$\begin{cases} \frac{\partial}{\partial t} \to -i\omega \\ \frac{\partial}{\partial \vec{x}} \to i\vec{k} \end{cases}$$

The Vlasov equation (63) becomes:

$$-i\omega\delta f_\alpha + i\vec{v}\cdot\vec{k}\delta f_\alpha + q_\alpha \delta E_\beta \frac{\partial f_\alpha}{\partial p_\beta} + \frac{1}{c}q_\alpha \left(\vec{v}\times\delta\vec{B}\right)_\beta \frac{\partial f_\alpha}{\partial p_\beta} + \frac{1}{c}q_\alpha \left(\vec{v}\times\vec{B}_0\right)_\beta \frac{\partial \delta f_\alpha}{\partial p_\beta} = 0 \tag{68}$$

Similarly Maxwell's equations read:

$$i\vec{k}\cdot\delta\vec{E} = 4\pi\zeta, \qquad i\vec{k}\cdot\delta\vec{B} = 0 \tag{69}$$

and

$$i\vec{k}\times\delta\vec{E} = i\frac{\omega}{c}\delta\vec{B}, \qquad i\vec{k}\times\delta\vec{B} = -i\frac{\omega}{c}\delta\vec{E} + \frac{4\pi}{c}\vec{J} \tag{70}$$

From (69) we already notice that the modes allowed to propagate are those perpendicular to the magnetic field perturbation, namely $\vec{k} \perp \delta\vec{B}$.
We now make a further simplifying assumption that B_0 is aligned along the $\hat{z}$ axis so that

$$\vec{B} = (0, 0, B_0) \tag{71}$$

Since $\vec{k} \perp \delta\vec{B}$, we can choose that $\vec{k} \parallel \vec{B}_0$, so

$$\vec{k} = (0, 0, k) \tag{72}$$

We can now perform the vector product in the first equation in (70), using the matrix notation:

$$i\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ \delta E_x & \delta E_y & \delta E_z \end{vmatrix} = i\frac{\omega}{c}\begin{pmatrix} \delta B_x \\ \delta B_y \\ \delta B_z \end{pmatrix} \tag{73}$$

from which we obtain the following conditions for the components:

$$\begin{cases} \delta B_x = -\frac{kc}{\omega}\delta E_y \\ \delta B_y = \frac{kc}{\omega}\delta E_x \\ \delta B_z = 0 \end{cases} \tag{74}$$

We obtained that there is no perturbation along $\vec{B}_0$. The only perturbative components are those perpendicular to the field $\vec{B}_0$, as expected.

We want now to turn back to the Vlasov equation in Fourier space, equation (68), and express all the quantities in terms of $\delta\vec{E}$

$$\begin{aligned}&-i\omega\delta f_\alpha + ikv_\parallel\delta f_\alpha + q_\alpha\left(\delta E_x\frac{\partial f_\alpha}{\partial p_x} + \delta E_y\frac{\partial f_\alpha}{\partial p_y}\right) + \\ &+\frac{q_\alpha}{c}\left[-v_\parallel\delta B_y\frac{\partial f_\alpha}{\partial p_x} + v_\parallel\delta B_x\frac{\partial f_\alpha}{\partial p_y} + (v_x\delta B_y - v_y\delta B_x)\frac{\partial f_\alpha}{\partial p_z}\right] + \\ &+\frac{q_\alpha}{c}\left[v_y B_0\frac{\partial\delta f_\alpha}{\partial p_x} - v_x B_0\frac{\partial\delta f_\alpha}{\partial p_y}\right] = 0\end{aligned} \tag{75}$$

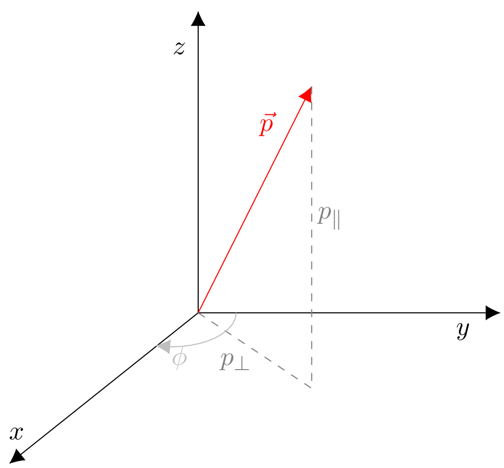
Substituting the expressions for δB_x and δB_y we obtain

$$\begin{aligned}&-i\omega\delta f_\alpha + ikv_\parallel\delta f_\alpha + q_\alpha\left(\delta E_x\frac{\partial f_\alpha}{\partial p_x} + \delta E_y\frac{\partial f_\alpha}{\partial p_y}\right) + \\ &+\frac{q_\alpha}{c}\left[-v_\parallel\frac{kc}{\omega}\delta E_x\frac{\partial f_\alpha}{\partial p_x} - v_\parallel\frac{kc}{\omega}\delta E_y\frac{\partial f_\alpha}{\partial p_y} + \frac{kc}{\omega}(v_x\delta E_x + v_y\delta E_y)\frac{\partial f_\alpha}{\partial p_z}\right] + \\ &+\frac{q_\alpha}{c}B_0\left[v_y\frac{\partial\delta f_\alpha}{\partial p_x} - v_x\frac{\partial\delta f_\alpha}{\partial p_y}\right] = 0\end{aligned} \tag{76}$$

At this point it is convenient to go to cylindrical coordinates

$$p_x, p_y, p_z \to p_\parallel, p_\perp, \phi \tag{77}$$

with ϕ the azimutal angle.



The transformations to pass to cylindrical coordinates are given by

$$\begin{cases}p_x = p_\perp\cos\phi \\ p_y = p_\perp\sin\phi \\ p_z = p_\parallel\end{cases} \implies \begin{cases}dp_x = dp_\perp\cos\phi - p_\perp\sin\phi dphi \\ dp_y = dp_\perp\sin\phi + p_\perp\cos\phi d\phi \\ dp_z = dp_\parallel\end{cases} \tag{78}$$

The inverse transformations are given by

$$\begin{cases}p_\perp = \sqrt{p_x^2 + p_y^2} \\ p_\parallel = p_z \\ \phi = \arctan\left(\frac{p_y}{p_x}\right)\end{cases} \implies \begin{cases}dp_\perp = \cos\phi dp_x + \sin\phi dp_y \\ dp_\parallel = dp_z \\ d\phi = -\frac{\sin\phi}{p_\perp}dp_x + \frac{\cos\phi}{p_\perp}dp_y\end{cases} \tag{79}$$

The derivatives of interest are

$$\begin{cases} \frac{\partial f_\alpha}{\partial p_x(p_\perp,p_\parallel,\phi)} = \frac{\partial f_\alpha}{\partial p_\perp}\frac{\partial p_\perp}{\partial p_x} + \frac{\partial f_\alpha}{\partial p_\parallel}\frac{\partial p_\parallel}{\partial p_x} + \frac{\partial f_\alpha}{\partial \phi}\frac{\partial \phi}{\partial p_x} = \cos\phi\frac{\partial f_\alpha}{\partial p_\perp} - \frac{\sin\phi}{p_\perp}\frac{\partial f_\alpha}{\partial \phi} \\ \frac{\partial f_\alpha}{\partial p_y(p_\perp,p_\parallel,\phi)} = \frac{\partial f_\alpha}{\partial p_\perp}\frac{\partial p_\perp}{\partial p_y} + \frac{\partial f_\alpha}{\partial p_\parallel}\frac{\partial p_\parallel}{\partial p_y} + \frac{\partial f_\alpha}{\partial \phi}\frac{\partial \phi}{\partial p_y} = \sin\phi\frac{\partial f_\alpha}{\partial p_\perp} + \frac{\cos\phi}{p_\perp}\frac{\partial f_\alpha}{\partial \phi} \\ \frac{\partial f_\alpha}{\partial p_z(p_\perp,p_\parallel,\phi)} = \frac{\partial f_\alpha}{\partial p_\perp}\frac{\partial p_\perp}{\partial p_z} + \frac{\partial f_\alpha}{\partial p_\parallel}\frac{\partial p_\parallel}{\partial p_z} + \frac{\partial f_\alpha}{\partial \phi}\frac{\partial \phi}{\partial p_z} = \frac{\partial f_\alpha}{\partial p_\parallel} \end{cases} \tag{80}$$

Therefore we can rewrite (76) and obtain:

$$\begin{aligned} &-i\omega\delta f_\alpha + ikv_\parallel \delta f_\alpha + q_\alpha\left(\delta E_x\cos\phi\frac{\partial f_\alpha}{\partial p_\perp} + \delta E_y \sin\phi\frac{\partial f_\alpha}{\partial p_\perp}\right) + \\ &+\frac{q_\alpha}{c}\left[-v_\parallel\frac{kc}{\omega}\delta E_x\cos\phi\frac{\partial f_\alpha}{\partial p_\perp} - v_\parallel\frac{kc}{\omega}\delta E_y\sin\phi\frac{\partial f_\alpha}{\partial p_\perp} + \frac{kc}{\omega}\left(v_\perp\cos\phi\delta E_x + v_\perp\sin\phi\delta E_y\right)\frac{\partial f_\alpha}{\partial p_\parallel}\right] + \\ &+\frac{q_\alpha}{c}B_0\left[v_\perp\sin\phi\left(\cos\phi\frac{\partial\delta f_\alpha}{\partial p_\perp} - \frac{\sin\phi}{p_\perp}\frac{\partial \delta f_\alpha}{\partial\phi}\right) - v_\perp\cos\phi\left(\sin\phi\frac{\partial\delta f_\alpha}{\partial p_\perp} + \frac{\cos\phi}{p_\perp}\frac{\partial\delta f_\alpha}{\partial\phi}\right)\right] = 0 \end{aligned} \tag{81}$$

where we have used $\partial f_\alpha/\partial\phi = 0$. We also make a further simplifying assumption that:

$$|\delta E_x| = |\delta E_y| = \delta E \implies \delta E_y = \pm i\delta E_x = \pm i\delta E \tag{82}$$

In other words the perturbation is circularly polarized.
At this point we can write:

$$-i\omega\delta f_\alpha + ikv_\parallel\delta f_\alpha + q_\alpha\delta E\frac{\partial f_\alpha}{\partial p_\perp}e^{\pm i\phi} - q_\alpha v_\parallel\frac{k}{\omega}\delta E\frac{\partial f_\alpha}{\partial p_\perp}e^{\pm i\phi} + q_\alpha v_\perp\frac{k}{\omega}\delta E\frac{\partial f_\alpha}{\partial p_\parallel}e^{\pm i\phi} - \frac{q_\alpha B_0 v_\perp}{cp_\perp}\frac{\partial\delta f_\alpha}{\partial\phi} = 0 \tag{83}$$

Setting $\delta f_\alpha = \delta g_\alpha e^{\pm i\phi}$, we obtain:

$$\delta g_\alpha\left(-i\omega + ikv_\parallel \pm i\Omega_\alpha\right) = q_\alpha\delta E\left[-\frac{\partial f_\alpha}{\partial p_\perp} + v_\parallel\frac{k}{\omega}\frac{\partial f_\alpha}{\partial p_\perp} - v_\perp\frac{k}{\omega}\frac{\partial f_\alpha}{\partial p_\parallel}\right] \tag{84}$$

where

$$\frac{q_\alpha B_0 v_\perp}{cm_\alpha\gamma v_\perp} = \frac{q_\alpha B_0}{m_\alpha c\gamma} \equiv \Omega_\alpha \tag{85}$$

with γ the Lorentz factor. Ω_α is the cyclotron frequency of the particles.
Finally we can write δf_α as:

$$\delta f_\alpha = \frac{\left\{\delta E e^{\pm i\phi}q_\alpha\left[-\frac{\partial f_\alpha}{\partial p_\perp} + \frac{k}{\omega}v_\parallel\frac{\partial f_\alpha}{\partial p_\perp} - \frac{k}{\omega}v_\perp\frac{\partial f_\alpha}{\partial p_\parallel}\right]\right\}}{-i\omega + ikv_\parallel \mp i\Omega_\alpha} \tag{86}$$

Observe that we have written δf_α as a function of δE only, so that everything is a function of δE: we have obtained our dispersion relation.
At this point we can take the Maxwell's equation $i\vec{k}\times\delta\vec{B} = -i\frac{\omega}{k}\delta\vec{E} + \frac{4\pi}{c}\vec{J}$, take the x component and substitute the expression for δf_α. First pass to Fourier space:

$$i\vec{k}\times\delta B = -i\frac{\omega}{c}\delta E + \frac{4\pi}{c}\vec{J} \tag{87}$$

Then:

$$i\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & k \\ \delta B_x & \delta B_y & 0 \end{vmatrix} = -i\frac{\omega}{c}\begin{pmatrix}\delta E_x \\ \delta E_y \\ 0\end{pmatrix} + \frac{4\pi}{c}\sum_\alpha\int d^3p q_\alpha\delta f_\alpha\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix} \tag{88}$$

The x component will be given by:

$$-ik\delta B_y = -i\frac{\omega}{c}\delta E_x + \frac{4\pi}{c}\sum_\alpha \int d^3p q_\alpha \delta f_\alpha v_x \tag{89}$$

where we can use the relation $\delta B_y = \frac{kc}{\omega}\delta E_x$ and the previously found expression for δf_α:

$$\frac{k^2c^2}{\omega^2} = 1 + 4\pi\frac{i}{\omega}\sum_\alpha \int d^3p q_\alpha v_\perp \cos\phi \frac{e^{\pm i\phi} q_\alpha \left[-\frac{\partial f_\alpha}{\partial p_\perp} + \frac{k}{\omega}v_\parallel \frac{\partial f_\alpha}{\partial p_\perp} - \frac{k}{\omega}v_\perp \frac{\partial f_\alpha}{\partial p_\parallel}\right]}{-i\omega + ikv_\parallel \mp i\Omega_\alpha} \tag{90}$$

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{4\pi}{\omega}\sum_\alpha q_\alpha^2 int d\phi \cos\phi e^{\pm i\phi} \int dp_\parallel dp_\perp p_\perp v_\perp \frac{\left[\frac{\partial f_\alpha}{\partial p_\perp} - \frac{k}{\omega}v_\parallel \frac{\partial f_\alpha}{\partial p_\perp} + \frac{k}{\omega}v_\perp \frac{\partial f_\alpha}{\partial p_\parallel}\right]}{\omega - kv_\parallel \pm \Omega_\alpha} \tag{91}$$

where we have used

$$\int d^3p = \int dp_\parallel \int dp_\perp p_\perp \int d\phi \tag{92}$$

The integral in ϕ is easily done

$$\int_0^{2\pi} d\phi e^{\pm i\phi}\cos\phi = \pi \tag{93}$$

So we are left with

$$\frac{k^2c^2}{\omega^2} = 1 + \sum_\alpha \frac{4\pi^2 q_\alpha^2}{\omega}\int dp_\parallel \int dp_\perp p_\perp v_\perp \frac{\frac{\partial f_\alpha}{\partial p_\perp}\left(1 - \frac{k}{\omega}v_\parallel\right) + \frac{k}{\omega}v_\perp \frac{\partial f_\alpha}{\partial p+\|}}{\omega - kv_\parallel \pm \Omega_\alpha} \tag{94}$$

We now move from $(p_\parallel, p_\perp)$ coordinates to (p, μ) coordinates, with $\mu = \cos\theta$, where:

$$\begin{cases} p_\parallel = p\mu \\ p_\perp = p(1-\mu^2)^{1/2} \end{cases} \implies \begin{cases} p = \sqrt{p_\parallel^2 + p_\perp^2} \\ \mu = \frac{p_\parallel}{\sqrt{p_\parallel^2 + p_\perp^2}} \end{cases} \tag{95}$$

with θ the angle between $\vec{v}$ and $\vec{B}_0$.

So that the derivatives of interest are:

$$\begin{cases} \frac{\partial f_\alpha}{\partial p_\parallel} = \frac{\partial f_\alpha}{\partial p}\frac{\partial p}{\partial p_\parallel} + \frac{\partial f_\alpha}{\partial \mu}\frac{\partial \mu}{\partial p_\parallel} = \mu\frac{\partial f_\alpha}{\partial p} + \frac{1-\mu^2}{p}\frac{\partial f_\alpha}{\partial \mu} \\ \frac{\partial f_\alpha}{\partial p_\perp} = \frac{\partial f_\alpha}{\partial p}\frac{\partial p}{\partial p_\perp} + \frac{\partial f_\alpha}{\partial \mu}\frac{\partial \mu}{\partial p_\perp} = (1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial p} - \frac{\mu}{p}(1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial \mu} \end{cases} \tag{96}$$

At last we need the measure of the integral:

$$\int dp_\parallel dp_\perp = dp d\mu p(1-\mu^2)^{-1/2} \tag{97}$$

where $p(1-\mu^2)^{-1/2}$ is the determinant of the Jacobian matrix, describing the coordinate transformations. Now we can elaborate on the numerator of the integrand in (94):

$$\begin{aligned}\frac{\partial f_\alpha}{\partial p_\perp}\left(1-\frac{kv_\parallel}{\omega}\right)+\frac{kv_\perp}{\omega}\frac{\partial f_\alpha}{\partial p+\parallel} &= \left[(1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial p}-\frac{\mu}{p}(1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial \mu}\right]\left(1-\frac{k}{\omega}v_\parallel\right)+\frac{k}{\omega}v_\perp\left[\mu\frac{\partial f_\alpha}{\partial p}+\frac{1-\mu^2}{p}\frac{\partial f_\alpha}{\partial \mu}\right]\\ &= \frac{\partial f_\alpha}{\partial p}\left[(1-\mu^2)^{1/2}\left(1-\frac{kv\mu}{\omega}\right)+\frac{kv\mu}{\omega}(1-\mu^2)^{1/2}\right]+\\ &\quad+\frac{\partial f_\alpha}{\partial \mu}\left[-\frac{\mu}{p}(1-\mu^2)^{1/2}\left(1-\frac{kv\mu}{\omega}\right)+\frac{kv}{\omega p}(1-\mu^2)^{3/2}\right]\\ &= (1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial p}+(1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial \mu}\left[-\frac{\mu}{p}+\frac{kv\mu^2}{p\omega}+\frac{kv}{\omega p}-\frac{kv\mu^2}{\omega p}\right]\\ &= (1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial p}+(1-\mu^2)^{1/2}\frac{\partial f_\alpha}{\partial \mu}\frac{1}{p}\left[-\mu+\frac{kv}{\omega}\right]\end{aligned} \tag{98}$$

Therefore our integral (94) becomes:

$$\frac{k^2c^2}{\omega^2}=1+\sum_\alpha\frac{4\pi^2q_\alpha^2}{\omega}\int dp\int d\mu\frac{p^2v(1-\mu^2)}{\omega-kv_\parallel\pm\Omega_\alpha}\left\{\frac{\partial f_\alpha}{\partial p}+\frac{1}{p}\frac{\partial f_\alpha}{\partial \mu}\left(\frac{kv}{\omega}-\mu\right)\right\} \tag{99}$$

We have obtained the general (though under some simplifying assumptions) expression for the allowed to propagate perturbations. It can also be written as

$$\frac{k^2c^2}{\omega^2}=1+\sum_{\alpha=i,e}\chi_\alpha \tag{100}$$

with χ_α the **susceptivity** of the plasma. Let us rewrite the dispersion relation

$$\frac{k^2c^2}{\omega^2}=1+\sum_{\alpha=i,e}\frac{4\pi^2q_\alpha^2}{\omega}\int dp\int d\mu\frac{p^2v(1-\mu^2)}{\omega-kv_\parallel\pm\Omega_\alpha}\left\{\frac{\partial f_\alpha}{\partial p}+\frac{1}{p}\frac{\partial f_\alpha}{\partial \mu}\left(\frac{kv}{\omega}-\mu\right)\right\} \tag{101}$$

All the quantities in the susceptivity depend on the unperturbed quantities.
We assume **cold plasma**, i.e. $\gamma=1$ and the momenta are approaching zero. So we can write

$$f_\alpha=A_\alpha\delta(p) \tag{102}$$

We can find A_α by imposing

$$\int dp4\pi p^2A_\alpha\delta(p)=n_\alpha \tag{103}$$

Therefore

$$A_\alpha=\frac{n_\alpha}{4\pi p^2} \tag{104}$$

- For electrons: $f_e=\frac{n_e}{4\pi p^2}\delta(p)$
- For ions: $f_p=\frac{n_p}{4\pi p^2}\delta(p)$

We note that there is no dependence on μ, which is good because the derivative part $\frac{\partial f_\alpha}{\partial \mu}$ will go to zero in (101). So only the first term in (101) remains, where we have a delta function though. We can integrate by parts (IBP):

$$\begin{aligned}
\frac{k^2c^2}{\omega^2} &= 1 + \frac{4\pi^2 q_\alpha^2}{\omega} \sum_{\alpha=e,p} \int dp \int d\mu \frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha} \frac{n_\alpha}{4\pi} \frac{\partial}{\partial p} \left(\frac{\delta(p)}{p^2} \right) \\
&= 1 + \frac{4\pi^2 q_\alpha^2}{\omega} \sum_{\alpha=e,p} \int d\mu \left[\cancel{\frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha} \frac{n_\alpha}{4\pi} \frac{\delta(p)}{p^2}} \Bigg|_0^{+\infty} - \int dp \frac{\delta(p)}{p^2} \frac{\partial}{\partial p} \left(\frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha} \frac{n_\alpha}{4\pi} \right) \right] \\
&= 1 - \frac{4\pi^2 e^2 n_p}{\omega} \int dp \int d\mu \frac{1}{4\pi p^2} \delta(p) \frac{\partial}{\partial p} \left(\frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_p} \right) + \\
&\quad - \frac{4\pi^2 e^2 n_e}{\omega} \int dp \int d\mu \frac{1}{4\pi p^2} \delta(p) \frac{\partial}{\partial p} \left(\frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_e} \right)
\end{aligned} \tag{105}$$

Let's look at the integral in p:

$$\begin{aligned}
\int dp \frac{1}{4\pi p^2} \delta(p) \frac{\partial}{\partial p} \left[\frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha} \right] &= \int dp \frac{1-\mu^2}{4\pi p^2} \delta(p) \left[\frac{\frac{3p^2}{m_\alpha}(\omega - kv\mu \pm \Omega_\alpha) + \frac{k\mu}{m_\alpha}\frac{p^3}{m_\alpha}}{(\omega - kv\mu \pm \Omega_\alpha)^2} \right] \\
&= (1-\mu^2) \frac{3}{4\pi m_\alpha} \frac{1}{\omega \pm \Omega_\alpha}
\end{aligned} \tag{106}$$

where we have used

$$v = \frac{p}{m_\alpha} \tag{107}$$

and have evaluated $\delta(p)$ which has set $p = 0$.
We are left with the integral in μ only

$$\int_{-1}^{+1} d\mu (1-\mu^2) \frac{3n_\alpha}{4\pi m_\alpha} \frac{1}{\omega \pm \Omega_\alpha} = \frac{n_\alpha}{\pi m_\alpha} \frac{1}{\omega \pm \Omega_\alpha} \tag{108}$$

We can now rewrite the initial expression (101) as follows:

$$\frac{k^2c^2}{\omega^2} = 1 - \frac{4\pi e^2 n_p}{\omega m_p} \frac{1}{\omega \pm \frac{eB_0}{m_p c}} - \frac{4\pi e^2 n_e}{\omega m_e} \frac{1}{\omega \mp \frac{eB_0}{m_e c}} \tag{109}$$

where we have used the explicit expression for the giration frequency Ω_α. We can collect the Ω_α term and obtain

$$\frac{k^2c^2}{\omega^2} = 1 \mp \frac{4\pi e^2 n_p}{\omega m_p} \frac{m_p c}{eB_0} \frac{1}{1 \pm \frac{\omega}{\Omega_p}} \pm \frac{4\pi e^2 n_e}{\omega m_e} \frac{m_e c}{eB_0} \frac{1}{1 \mp \frac{\omega}{\Omega_e}} \tag{110}$$

Usually we are interested in phenomena involving a large number of particles, so it is sensible to take the low frequency limit of the perturbations, which means that

$$\omega \ll \Omega_p \ll |\Omega_e| \tag{111}$$

Therefore we can Taylor expand (110) and we end up with

$$\frac{k^2c^2}{\omega^2} = 1 \mp \frac{4\pi e^2 n_p}{\omega m_p} \underbrace{\frac{m_p c}{eB_0}}_{\frac{1}{\Omega_p}} \left(1 \mp \frac{\omega}{\Omega_p} \right) \pm \frac{4\pi e^2 n_{p(=e)}}{\omega m_{p(=e)}} \frac{m_{p(=e)} c}{eB_0} \tag{112}$$

where we have used the fact that at equilibrium $n_e = n_p$ (charge neutrality) and the fact that the electron mass m_e cancels out.

$$\begin{aligned}\frac{k^2c^2}{\omega^2} &= 1 + \frac{4\pi e^2 n_0}{m_p \cancel{\omega}} \frac{\cancel{\omega}}{\Omega_p^2} \\ &= 1 + \frac{4\pi \cancel{e^2} n_0}{\cancel{m_p}} \frac{m_p^{\cancel{2}} c^2}{\cancel{e^2} B_0^2} \\ &= 1 + \underbrace{\frac{4\pi n_0 m_p}{B_0^2}}_{\left[\frac{1}{v^2}\right]} c^2\end{aligned} \tag{113}$$

For dimensional arguments the factor multiplying c^2 should be an inverse of a velocity, we call it **Alfven speed**

$$v_A = \frac{B_0}{\sqrt{4\pi n_0 m_p}} \tag{114}$$

So we can finally write

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{c^2}{v_A^2} \tag{115}$$

For typical conditions in the interstellar medium v_A is of order of few km/s. Therefore

$$\frac{c^2}{v_A^2} \gg 1 \tag{116}$$

so we can ignore the 1 term and write our dispersion relation as

$$\frac{k^2\cancel{c^2}}{\omega^2} = \frac{\cancel{c^2}}{v_A^2} \implies \omega^2 = k^2 v_A^2 \implies \omega = \pm k v_A \tag{117}$$

We have what are called Alfven waves, which move at slow velocities and are crucial for thermal particles. Non thermal particles are able themselves to produce Alfven waves.

Summing up ...

The particle distribution function f_α evolves within a plasma following the Vlasov equation:

$$\frac{\partial f_\alpha}{\partial t} + \vec{v} \cdot \vec{\nabla} f_\alpha + q_\alpha \left[\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right]_\beta \frac{\partial f_\alpha}{\partial p_\beta} = 0$$

where $\vec{E}$ and $\vec{B}$ represent the collective electric and magnetic fields.
In we assume only a large scale magnetic field $\vec{B}_0$ aligned along the z axis and perturb the system we obtain the dispersion relation describing the modes allowed to propagate:

$$\frac{k^2c^2}{\omega^2} = 1 + \sum_\alpha \frac{4\pi^2 q_\alpha^2}{\omega} \int dp \int d\mu \frac{p^2 v(1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha} \left[\frac{\partial f_\alpha}{\partial p} + \frac{1}{p}\frac{\partial f_\alpha}{\partial \mu}\left(\frac{kv}{\omega} - \mu\right)\right]$$

Under the hypothesis of **cold plasma** (Lorentz factor $\gamma = 1$) we can write the distribution function as $f_\alpha = \frac{n_\alpha}{4\pi p^2}\delta(p)$, so that the dispersion relation simplifies to

$$\omega^2 = k^2 v_A^2$$

with $v_A = \frac{B_0}{\sqrt{4\pi n_0 m_p}}$ the Alfven velocity.

IV. Ideal MagnetoHydroDynamics (MHD) equations

We would like now to deal with quantities which are more practical and we will start from the Vlasov equation, which describes a set of particles behaving in a collisionless way. We will use the non relativistic version

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_x f + \frac{q}{m}\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right] \cdot \vec{\nabla}_v f = 0 \tag{118}$$

Remember that if we have a generic function $\phi(\vec{x}, \vec{v}, t)$ we can define its mean value as

$$\begin{aligned} \langle \phi(\vec{x}, \vec{v}, t) \rangle_V &= \frac{\int d^3\vec{v} \phi(\vec{x}, \vec{v}, t) f(\vec{x}, \vec{v}, t)}{\int d^3\vec{v} f(\vec{x}, \vec{v}, t)} \\ &= \frac{1}{n(\vec{x}, t)} \int d^3\vec{v} \phi(\vec{x}, \vec{v}, t) f(\vec{x}, \vec{v}, t) \end{aligned} \tag{119}$$

Now suppose that ψ is function of the velocity only

$$\psi = \psi(\vec{v}) \tag{120}$$

we can multiply all the terms in the Vlasov equation (118) by $\psi(\vec{v})$ and then integrate in $d^3\vec{v}$. Let's analyze each term separately

1. $$\int d^3\vec{v} \psi(\vec{v}) \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int d^3\vec{v} \psi f = \frac{\partial}{\partial t}\left[n\langle \psi \rangle\right] \tag{121}$$

2. $$\int d^3\vec{v} \psi v_r \frac{\partial f}{\partial x_r} = \frac{\partial}{\partial x_r} \int d^3\vec{v} \psi(\vec{v}) v_r f = \frac{\partial}{\partial x_r}\left[n\langle \psi v_r \rangle\right] \tag{122}$$

3. $$\begin{aligned} \int d^3\vec{v} \psi(\vec{v}) \frac{q}{m}\left[E_r + \epsilon_{rst}\frac{v_s}{c}B_t\right] \cdot \frac{\partial f}{\partial v_r} &= \frac{q}{m}E_r \underbrace{\int d^3\vec{v} \psi(\vec{v}) \frac{\partial f}{\partial v_r}}_{IBP} + \frac{q}{mc}\epsilon_{rst}B_t \underbrace{\int d^3\vec{v} \psi(\vec{v}) v_s \frac{\partial f}{\partial v_r}}_{IBP} \\ &= -\frac{q}{m}E_r n\langle \frac{\partial \psi}{\partial v_r} \rangle - \frac{q}{mc}\epsilon_{rst}B_t \int d^3\vec{v} f \cdot \frac{\partial}{\partial v_r}\left(\psi v_s\right) \\ &= -\frac{q}{m}E_r n\langle \frac{\partial \psi}{\partial v_r} \rangle - \frac{q}{mc}\epsilon_{rst}B_t \int d^3\vec{v} f \left(\frac{\partial \psi}{\partial v_r} v_s + \psi \underbrace{\delta_{rs}}_{=0}\right) \\ &= -\frac{q}{m}E_r n\langle \frac{\partial \psi}{\partial v_r} \rangle - \frac{q}{mc}\epsilon_{rst}B_t n\langle \frac{\partial \psi}{\partial v_r} \cdot v_s \rangle \end{aligned} \tag{123}$$

   where the δ_{rs} term goes to zero because of the antisymmetric Levi-Civita tensor.

- If we choose ψ constant, the last term goes to zero

  $$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_r}\left[n u_r\right] = 0 \tag{124}$$

  where we have defined

  $$u_r = \langle v_r \rangle \tag{125}$$

  This resembles the continuity equation, which expresses the conservation of mass.

- If we choose $\psi = v_r$ we obtain

$$\frac{\partial}{\partial t}\left[n u_r\right] + \frac{\partial}{\partial x_s}\left[n\langle v_r v_s\rangle\right] - \frac{qnE_r}{m} - \frac{qn}{mc}\epsilon_{rst}B_t u_s = 0 \tag{126}$$

  If we now multiply everything by m ($mn = \rho$), we obtain

$$\frac{\partial}{\partial t}\left[\rho u_r\right] + \frac{\partial}{\partial x_s}\left[\rho\langle v_r v_s\rangle\right] - qnE_r - \frac{qn}{c}\epsilon_{rst}B_t u_s = 0 \tag{127}$$

Remember that we should sum over the species number. Therefore it would be more useful to have electrons and protons expressed as a single fluid, in the following way:

1. $\rho = n_p m_p + n_e m_e$
2. $\zeta = (n_p - n_e)e$
3. $u_{p,r} = \langle v_{p,r}\rangle$, $u_{e,r} = \langle v_{e,r}\rangle$
   We can define the velocity of the center of mass as

$$U_r = \frac{n_p m_p u_{p,r} + n_e m_e u_{e,r}}{n_p m_p + n_e m_e} \tag{128}$$

4. Current component

$$J_r = (n_p u_{p,r} - n_e u_{e,r})e \tag{129}$$

5. Let's also define a new object w, representing the mean relative velocity with respect to the center of mass velocity U (this represents fluctuations, so temperature and therefore pressure):

$$\begin{cases} \langle w_{p,r}\rangle = u_{p,r} - U_r \\ \langle w_{e,r}\rangle = u_{e,r} - U_r \end{cases} \tag{130}$$

   It is associated with the internal motion of particles

6. Introduce also the pressure:

$$\begin{aligned} P_{p,rs} &= m_p \int d^3\vec{v} f_p w_{p,r} w_{p,s} \equiv m_p n_p \langle w_{p,r} w_{p,s}\rangle \\ P_{e,rs} &= m_e \int d^3\vec{v} f_e w_{e,r} w_{e,s} \equiv m_e n_e \langle w_{e,r} w_{e,s}\rangle \end{aligned} \tag{131}$$

Let's go back to the conservation of mass equation

- For protons:

$$\frac{\partial}{\partial t}\left[n_p m_p\right] + \frac{\partial}{\partial x_r}\left[m_p n_p u_{p,r}\right] = 0 \tag{132}$$

- For electrons:

$$\frac{\partial}{\partial t}\left[n_e m_e\right] + \frac{\partial}{\partial x_r}\left[m_e n_e u_{e,r}\right] = 0 \tag{133}$$

If we sum up the two, we end up with the **global equation of mass conservation**:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}\left[\rho U_r\right] = 0 \tag{134}$$

We can do the same for (127)

- For protons:

$$\frac{\partial}{\partial t}\left[\rho_p u_{p,r}\right]+\frac{\partial}{\partial x_s}\left[\rho_p\langle v_{p,r}v_{p,s}\rangle\right]-en_pE_r-\frac{en_p}{c}\epsilon_{rst}B_tu_{p,s}=0 \tag{135}$$

- For electrons:

$$\frac{\partial}{\partial t}\left[\rho_e u_{e,r}\right]+\frac{\partial}{\partial x_s}\left[\rho_e\langle v_{e,r}v_{e,s}\rangle\right]-en_eE_r-\frac{en_e}{c}\epsilon_{rst}B_tu_{e,s}=0 \tag{136}$$

Remember that $w_r = v_r - U_r$, so that $v_r = w_r + U_r$ and

$$\begin{aligned}\langle v_rv_s\rangle &= \langle (w_r+U_r)(w_s+U_s)\rangle\\ &= \langle w_rw_s\rangle + \langle w_r\rangle U_s + U_r\langle w_s\rangle + U_rU_s\\ &= \frac{P_{rs}}{nm}+u_rU_s+U_ru_s-U_rU_s\end{aligned} \tag{137}$$

where **pressure** is defined as[7]

$$P_{rs} = nm\langle w_rw_s\rangle \tag{138}$$

We can now sum the two equations, (135) and (136)

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho U_r\right]+\frac{\partial}{\partial x_s}\left[P_{p,rs}+\rho_pu_{p,r}U_s+\rho_pu_{p,s}U_r-\rho_pU_rU_s+P_{e,rs}+\rho_eu_{e,r}U_s+\rho_eu_{e,r}U_r-\rho_eU_rU_s\right]+\\ &-\zeta E_r-\frac{1}{c}\epsilon_{rst}B_tJ_s=0\end{aligned} \tag{139}$$

Therefore

$$\frac{\partial}{\partial t}\left[\rho U_r\right]+\frac{\partial}{\partial x_s}\left[P_{rs}+\rho U_rU_s\right]-\zeta E_r-\frac{1}{c}\epsilon_{rst}B_tJ_s=0 \tag{140}$$

Let's concentrate on the first two terms and use continuity equation for $\frac{\partial\rho}{\partial t}$

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho U_r\right]+\frac{\partial}{\partial x_s}\left[P_{rs}+\rho U_rU_s\right] &= \frac{\partial\rho}{\partial t}U_r+\rho\frac{\partial U_r}{\partial t}+\frac{\partial P_{rs}}{\partial x_s}+\frac{\partial(\rho U_s)\cdot U_r}{\partial x_s}+\rho U_s\frac{\partial U_r}{\partial x_s}\\ &= \rho\left[\frac{\partial}{\partial t}+U\cdot\nabla\right]U_r+\frac{\partial P_{rs}}{\partial x_s}\end{aligned} \tag{141}$$

So we can write

$$\rho\frac{DU_r}{Dt}=-\frac{\partial P_{rs}}{\partial x_s}+\zeta E_r+\frac{1}{c}\epsilon_{rst}B_tJ_s \tag{142}$$

The LHS is nothing but the Newton force, then on the RHS we have, in order, the gradient of the pressure, the electric force and the Lorentz force.
If we give up with the description of individual species we obtain that

1. mass is conserved

2. momentum is conserved

This is the basic formulation of MHD. We will assume an ideal MHD, which means that the **conductivity will be infinite**, i.e. there won't be electric fields. This is the case of plasmas, where it is extremely difficult to have electric fields.
Nevertheless we observe thermal particles, even if there are no electric fields to accelerate them (and magnetic fields do no work). How is this possible? We will soon deal with this problem.

[7] Remember also that $\langle w_r\rangle = u_r - U_r$

Suppose we do NOT have an electric field, but we still have to figure out what the current $\vec{J}$ is in this system. Suppose we have an electric field in the lab $\vec{E}$. In a comoving reference frame this will be given by $\vec{E}'$ thanks to the following generalization of the Ohm's law:

$$\vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{B} = \eta \vec{J} \tag{143}$$

η is the resistivity of the plasma. If resistivity η approaches zero, then the electric field is the induced one only:

$$\vec{E} = -\frac{v}{c} \times \vec{B} \tag{144}$$

We can now use the Maxwell's equation $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ and invert it to obtain

$$\vec{J} = \frac{c}{4\pi}\left[\vec{\nabla} \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t}\right] \tag{145}$$

so that we can express the vector product $\vec{J} \times \vec{B}$ in a closed way

$$\frac{1}{c}\vec{J} \times \vec{B} = \frac{1}{4\pi}\left\{\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \times \vec{B}\right\} \tag{146}$$

We want now to show that the second term in parenthesis is negligible with respect to the first one. To estimate the typical order of magnitude of the quantities in our system, we call L the scale at which the system changes dynamically (we have gradients). Therefore:

$$\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} \sim \frac{B^2}{L} \tag{147}$$

and

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} \times \vec{B} \sim \frac{1}{c}\frac{E}{T}B \sim \frac{1}{c}EB\frac{V}{L} \sim \frac{1}{c}\frac{vB}{c}B\frac{v}{L} \sim \frac{v^2}{c^2}\frac{B^2}{L} \tag{148}$$

where we have used

$$\vec{E} \sim -\frac{\vec{v}}{c} \times \vec{B} \implies E \sim \frac{v}{c}B \tag{149}$$

The second term in (146) is negligible with respect to the first, so we can say that

$$\frac{1}{c}\vec{J} \times \vec{B} = \frac{1}{4\pi}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} \tag{150}$$

Therefore we can write the two equations of ideal MHD as

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{151}$$

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{v}\right] = -\vec{\nabla}P + \frac{1}{4\pi}\left(\vec{\nabla} \times \vec{B}\right) \times \vec{B} \tag{152}$$

where we have renamed the center of mass velocity $\vec{U}$ with $\vec{v}$.

iv.1 Flux freezing

If a plasma is regulated by ideal MHD, then the flux of the magnetic field through a surface moving through the plasma has to be conserved. If $\eta = 0$ then the electric field in the lab is the induced only $\vec{E} = -\frac{1}{c}\vec{v} \times \vec{B}$. Let's define a flux Φ_0

$$\Phi_0 = \int d\vec{S}_0 \vec{B}(\vec{x}, t) \tag{153}$$

When the plasma moves we get

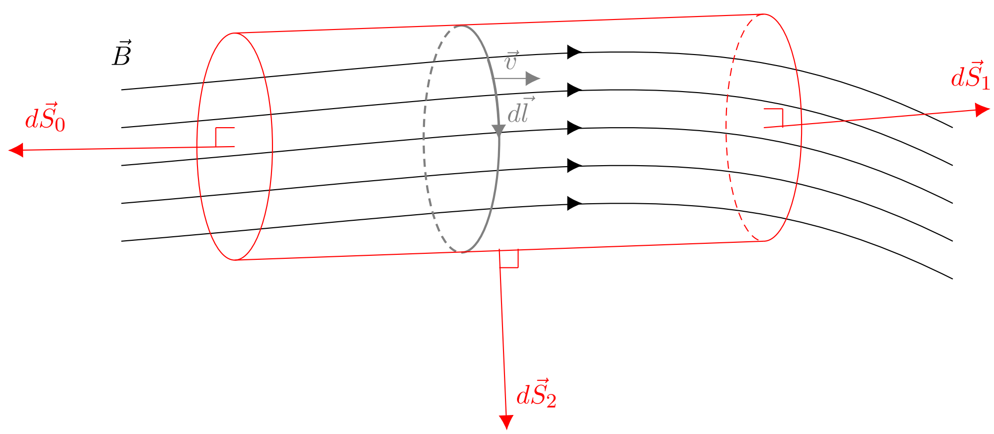
$$\Phi_1 = \int d\vec{S}_1 \vec{B}(\vec{x} + \vec{v}dt, t + dt) \tag{154}$$

What happens to the evolution of flux as a function of time? We can use Taylor expansion

$$\begin{aligned}\Phi_1 &= \int_{S_1} d\vec{S}_1 \vec{B}(\vec{x} + \vec{v}dt, t + dt) \\ &= \int_{S_1} d\vec{S}_1 \left[\vec{B}(\vec{x} + \vec{v}dt, t) + dt\frac{\partial \vec{B}}{\partial t}(\vec{x} + \vec{v}dt, t)\right] \\ &= \int_{S_1} d\vec{S}_1 \vec{B}(\vec{x} + \vec{v}dt, t) + dt\int_{S_0} d\vec{S}_0 \frac{\partial \vec{B}}{\partial t}(\vec{x} + \vec{v}dt, t)\end{aligned} \tag{155}$$

We can use the Gauss' theorem, through the whole closed surface we have that:

$$\int_S d\vec{S} \cdot \vec{B} = \int dV(\vec{\nabla} \cdot \vec{B}) \tag{156}$$



$$\int_S \vec{B} \cdot \vec{S} = -\underbrace{\int d\vec{S}_0 \vec{B}}_{\Phi_0} + \int d\vec{S}_1 \vec{B} + \int d\vec{S}_2 \vec{B} \tag{157}$$

So

$$\Phi_0 = \int d\vec{S}_1 \vec{B} + \int d\vec{S}_2 \vec{B} \tag{158}$$

We can also say:

$$\int d\vec{S}_1 \cdot \vec{B}(\vec{x} + \vec{v}dt, t) = \Phi_0 - \int d\vec{S}_2 \cdot \vec{B} \tag{159}$$

We can now use the equation for Φ_1

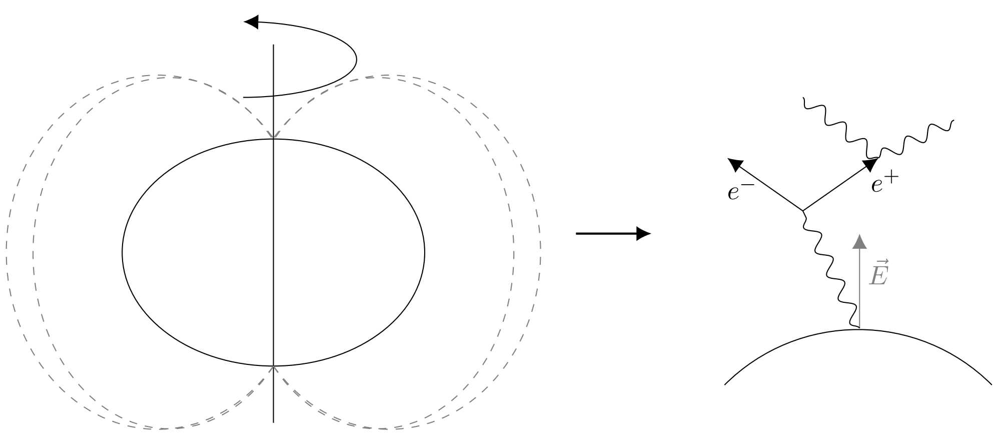
$$\Phi_1 = \Phi_0 - \int_{S_2} d\vec{S}_2 \cdot \vec{B} + \underbrace{dt\int_{S_0} d\vec{S}_0 \cdot \frac{\partial \vec{B}}{\partial t}}_{\text{perturbation term}} \tag{160}$$

We can calculate the surface element $d\vec{S}_2$ as $d\vec{S}_2 = d\vec{l} \times \vec{v}dt$
Using the property $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$ and the Stokes' theorem

$$\begin{aligned}\Phi_1 &= \Phi_0 - \int \vec{B} \cdot (d\vec{l} \times \vec{v}dt) + dt \int d\vec{S}_0 \cdot \frac{\partial \vec{B}}{\partial t} \\ &= \Phi_0 - dt \oint_{\partial S_1} (\vec{v} \times \vec{B}) \cdot d\vec{l} + dt \int d\vec{S}_0 \cdot \frac{\partial \vec{B}}{\partial t} \\ &= \Phi_0 - dt \int_{S_1} (\vec{\nabla} \times (\vec{v} \times \vec{B})) \cdot d\vec{S}_0 + dt \int_{S_0} d\vec{S}_0 \cdot \frac{\partial \vec{B}}{\partial t} \\ &= \Phi_0 + dt \int_{S_0} d\vec{S}_0 \left[\frac{\partial \vec{B}}{\partial t} - \underbrace{\vec{\nabla} \times (\vec{v} \times \vec{B})}_{\frac{\partial \vec{B}}{\partial t}} \right] = \Phi_0\end{aligned} \tag{161}$$

The flux calculated across the surface moving through the plasma is conserved: the magnetic lines reorganize themselves. Since the flux remains constant in time for any arbitrary surface, this implies that the magnetic field lines must move with the plasma. This is to say that the magnetic field lines are *frozen* into the plasma. Magnetic field lines embedded in an ideal MHD plasma can never break and reconnect.
Remember that we have assumed $\eta = 0$ to obtain these results. To break ideal MHD, we need a non negligible resistivity. There are situations in which this happens:

- **Magnetic ring connection** phenomena: when magnetic field lines are directed towards each other and $\eta \neq 0$, $\vec{B}$ can dissipate into $\vec{E}$ field, creating heating and kinetic energy (*solar flares*).
- **Surface of a neutron star**: here we have dipolar magnetic fields that can induce electric fields. Close to the surface we can break ideal MHD and have $\vec{E} \parallel \vec{B}$. We can think of a *gedanken experiment* imagining a neutron star in vacuum and asking ourselves whether it is possible. A neutron star spins around and its rotation induces an electric field on the surface which is larger that the gravitational force and directed outwards so that electrons are pushed out. Electrons can in turn radiate photons which can interact and produce pairs of electron and positrons leading to thousands of particles. Therefore we cannot have a spinning neutron star in vacuum.



iv.2 Perturbations in ideal MHD

What are the perturbations allowed in this ideal MHD? Here we list the relevant equations and then perturb them

1. The mass conservation equation
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{162}$$

2. The momentum conservation equation:
$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} P + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B} \tag{163}$$

3. The adiabaticity condition which ensures that entropy is constant (we use $\gamma = 5/3$):
$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}\right] P \rho^{-\gamma} = 0 \tag{164}$$

4. The induction equation:
$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) \tag{165}$$

5. The Maxwell's equation:
$$\vec{\nabla} \cdot \vec{B} = 0 \tag{166}$$

Our system possesses an ordered magnetic field $\vec{B}_0$ which we can assume to be aligned along the z direction
$$\vec{B}_0 \equiv (0, 0, B_0) \tag{167}$$

Let's perturb the equations above in Fourier space, remembering that we put ourselves in the reference frame of the plasma so that $\vec{v} = \delta \vec{v}$ and we discard second order perturbations or larger.

1. $$- i\omega \delta \rho + i \rho \vec{k} \cdot \delta \vec{v} = 0 \tag{168}$$

2. $$- i\omega \rho \delta \vec{v} = -i \vec{k} \delta P + \frac{i}{4\pi} \left(\vec{k} \times \delta \vec{B}\right) \times \vec{B}_0 \tag{169}$$

3. $$- i\omega \delta P \rho^{-\gamma} + i\omega P \gamma \rho^{-\gamma-1} \delta \rho = 0 \tag{170}$$

   That implies
   $$\frac{\delta P}{\delta \rho} = \frac{\gamma P}{\rho} \equiv c_s^2 \tag{171}$$

   where c_s^2 is the square of the sound speed. Therefore we can rewrite (169) using both (168) and (170)
   $$\begin{cases} \delta P = c_s^2 \delta \rho \\ \delta \rho = \frac{\rho}{\omega} \vec{k} \cdot \delta \vec{v} \end{cases} \tag{172}$$
   $$\implies \delta P = c_s^2 \frac{\rho}{\omega} \vec{k} \cdot \delta \vec{v} \tag{173}$$

   Remember that we want to rewrite all the perturbations in terms of one only

4. $$- i\omega \delta \vec{B} = i \vec{k} \times \left(\delta \vec{v} \times \vec{B}_0\right) \implies \delta \vec{B} = -\frac{1}{\omega} \vec{k} \times \left(\delta \vec{v} \times \vec{B}_0\right) \tag{174}$$

5. $$\vec{k} \perp \delta \vec{B} \iff \vec{k} \cdot \delta \vec{B} = 0 \tag{175}$$

Now we replace everything in equation (169) and get

$$-i\omega\rho\delta\vec{v} = -i\vec{k}c_s^2\frac{\rho}{\omega}\vec{k}\cdot\delta\vec{v} - \frac{i}{4\pi}\frac{1}{\omega}\left[\vec{k}\times\left(\vec{k}\times\left(\delta\vec{v}\times\vec{B}_0\right)\right)\right]\times\vec{B}_0 \tag{176}$$

This is the **dispersion relation** we were looking for. We shall now put ourselves in a particular setup and solve the vector products:

$$\begin{gathered}\vec{k}\parallel\vec{B}_0 \implies \vec{B}_0\equiv(0,0,B_0) \implies \vec{k}=(0,0,k)\\ \vec{\nabla}\times\vec{B}=0 \implies \vec{k}\cdot\delta\vec{B}=0 \implies \delta\vec{B}\equiv(\delta B_x,\delta B_y,0)\end{gathered} \tag{177}$$

We are now to solve the nested vector products:

- $$\delta\vec{v}\times\vec{B}_0 = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \delta v_x & \delta v_y & \delta v_z\\ 0 & 0 & B_0\end{vmatrix} = \begin{pmatrix}B_0\delta v_y\\ -B_0\delta v_x\\ 0\end{pmatrix} \tag{178}$$

- $$\vec{k}\times\left(\delta\vec{v}\times\vec{B}_0\right) = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & k\\ B_0\delta v_y & -B_0\delta v_x & 0\end{vmatrix} = \begin{pmatrix}kB_0\delta v_x\\ kB_0kv_y\\ 0\end{pmatrix} \tag{179}$$

- $$\vec{k}\times\left[\vec{k}\times\left(\delta\vec{v}\times\vec{B}_0\right)\right] = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & k\\ kB_0\delta v_y & kB_0\delta v_y & 0\end{vmatrix} = \begin{pmatrix}-k^2B_0\delta v_y\\ k^2B_0\delta v_x\\ 0\end{pmatrix} \tag{180}$$

- $$\left[\vec{k}\times\left[\vec{k}\times\left(\delta\vec{v}\times\vec{B}_0\right)\right]\right]\times\vec{B}_0 = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ -k^2B_0\delta v_y & k^2B_0\delta v_x & 0\\ 0 & 0 & B_0\end{vmatrix} = \begin{pmatrix}k^2B_0^2\delta v_x\\ -k^2B_0^2\delta v_y\\ 0\end{pmatrix} \tag{181}$$

Therefore we can write the perturbations as

$$\begin{pmatrix}\delta v_x\\ \delta v_y\\ \delta v_z\end{pmatrix} = \frac{c_s^2}{\omega^2}k^2\begin{pmatrix}0\\ 0\\ \delta v_z\end{pmatrix} + \frac{k^2B_0^2}{4\pi\rho\omega^2}\begin{pmatrix}\delta v_x\\ \delta v_y\\ 0\end{pmatrix} \tag{182}$$

$$\implies \begin{pmatrix}\delta v_x\\ \delta v_y\end{pmatrix} = \frac{k^2B_0^2}{4\pi\rho\omega^2}\begin{pmatrix}\delta v_x\\ \delta v_y\end{pmatrix} \implies \omega^2 = \frac{k^2B_0^2}{4\pi\rho} \tag{183}$$

Remember that the Alfven velocity is $v_A = \frac{B_0}{\sqrt{4\pi\rho}}$, therefore

$$\omega^2 = k^2v_A^2 \tag{184}$$

We have found again the Alfven modes, besides the sound waves

$$\delta v_z = \frac{c_s^2}{\omega^2}k^2\delta v_z \implies \omega^2 = c_s^2k^2 \tag{185}$$

So, perturbing a system like this gives us, when $\vec{k}\parallel\vec{B}_0$, two types of waves:

1. sound waves: move along B_0
2. Alfven waves: move in the direction perpendicular to B_0

We still have the induced electric field, which is in the xy plane, perpendicular to both $\vec{B}_0$ and $\delta\vec{v}$. For Alfven waves $\delta\vec{B} \equiv (\delta B_x, \delta B_y, 0)$ and therefore $\delta\vec{E} = \frac{1}{c}\delta\vec{v} \times \vec{B}_0$, which has the strength of order of v_A/cB_0, which is rather small. Nevertheless, even if its mean value is zero, it can still affect the mean value of the energy of a particle.

We analyzed previously the case when $\vec{k} \parallel \vec{B}_0$, now we are going to analyze the case when $\vec{k} \perp \vec{B}_0$, under the following assumptions

$$\vec{k} \perp \vec{B}_0, \quad \vec{B}_0 \equiv (0, 0, B_0), \quad \vec{k} = (k, 0, 0) \tag{186}$$

Doing the nested vector products we obtain:

- $$\delta\vec{v} \times \vec{B}_0 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \delta v_x & \delta v_y & \delta v_z \\ 0 & 0 & B_0 \end{vmatrix} = \begin{pmatrix} B_0 \delta v_y \\ -B_0 \delta v_x \\ 0 \end{pmatrix} \tag{187}$$
- $$\vec{k} \times \left(\delta\vec{v} \times \vec{B}_0\right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k & 0 & 0 \\ B_0\delta v_y & -B_0\delta v_x & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -kB_0\delta v_x \end{pmatrix} \tag{188}$$
- $$\vec{k} \times \left[\vec{k} \times \left(\delta\vec{v} \times \vec{B}_0\right)\right] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k & 0 & 0 \\ 0 & 0 & -kB_0\delta v_x \end{vmatrix} = \begin{pmatrix} 0 \\ k^2 B_0 \delta v_x \\ 0 \end{pmatrix} \tag{189}$$
- $$\left[\vec{k} \times \left[\vec{k} \times \left(\delta\vec{v} \times \vec{B}_0\right)\right]\right] \times \vec{B}_0 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & k^2 B_0 \delta v_x & 0 \\ 0 & 0 & B_0 \end{vmatrix} = \begin{pmatrix} k^2 B_0^2 \delta v_x \\ 0 \\ 0 \end{pmatrix} \tag{190}$$

Therefore:

$$-i\omega\rho \begin{pmatrix} \delta v_x \\ \delta v_y \\ \delta v_z \end{pmatrix} = -ik^2 c_S^2 \frac{\rho}{\omega} \begin{pmatrix} \delta v_x \\ 0 \\ 0 \end{pmatrix} - \frac{i}{4\pi}\frac{1}{\omega} \begin{pmatrix} k^2 B_0^2 \delta v_x \\ 0 \\ 0 \end{pmatrix} \tag{191}$$

We obtain the following final result:

$$\begin{pmatrix} \delta v_x \\ \delta v_y \\ \delta v_z \end{pmatrix} = \frac{c_S^2 k^2}{\omega^2} \begin{pmatrix} \delta v_x \\ 0 \\ 0 \end{pmatrix} + k^2 \underbrace{\frac{B_0^2}{4\pi\rho}}_{v_A^2} \begin{pmatrix} \delta v_x \\ 0 \\ 0 \end{pmatrix} \tag{192}$$

The transported perturbations are parallel to $\hat{x}$ and satisfy

$$v^2 = c_S^2 + v_A^2 \tag{193}$$

These are called the **magnetosonic modes** and they are embedded in the structure of ideal MHD.
So a charged particle in a plasma will experience the following motions:

- helicoidal motion for $\vec{B}_0$
- motion because of $\delta\vec{B}$
- interaction with the induced electric field $\vec{E}$

Ideal MHD sum up

The equations of ideal MHD are based on:

- **Conservation of mass**:
$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$
- **Conservation of momentum**:
$$\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} \vec{P} + \frac{1}{4\pi} \left(\vec{\nabla} \times \vec{B} \right) \times \vec{B}$$
- **Adiabaticity** (conservation of energy):
$$\left[\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right] \left(P \rho^{-\gamma} \right) = 0$$
- **Induction equation** (infinite conductivity): so that the electric field is the induced one only:
$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right)$$
  The hypothesis of infinite conductivity is the one that leads to the **flux freezing** effect.
- **Maxwell's equation** for the divergence of the magnetic field:
$$\vec{\nabla} \cdot \vec{B} = 0$$

Perturbing the equations above, we find that the perturbations allowed to propagate in a plasma governed by ideal MHD are (in the hypothesis that the large scale magnetic field is aligned along the z axis):

- $\vec{k} \parallel \vec{B}_0$: we have **sound waves** moving along $\vec{B}_0$ with velocity c_S and **Alfven waves** in the direction perpendicular to $\vec{B}_0$.
- $\vec{k} \perp \vec{B}_0$, ($\vec{k} \parallel \hat{x}$): we have the so called **magnetosonic waves** moving with velocity $v^2 = c_S^2 + v_A^2$.

II. Basics of transport of charged particles

i. Motion of a charged particle in an ordered magnetic field $\vec{B}_0$

We are going to throw a charged particle, already superthermal, in the plasma system and analyze what happens, ignoring the perturbations on small scales. The plasma possesses an ordered magnetic field of its own

$$\vec{B}_0 \equiv (0, 0, B_0) \tag{194}$$

We are going to use a relativistic approach to calculate the unperturbed trajectory (no electric field, just the Lorentz force):

$$\vec{p} = m\vec{v}\gamma \tag{195}$$

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v} \times \vec{B}_0 \implies m\gamma\frac{d\vec{v}}{dt} = \frac{q}{c}\vec{v} \times \vec{B}_0 \tag{196}$$

If there are no energy losses, the Lorentz force remains constant. The energy of the particle cannot change, since $\vec{B}_0$ cannot perform any work.
Let us analyze the single components:

$$\begin{cases} m\gamma\frac{dv_x}{dt} = \frac{q}{c}v_y B_0 \implies m\gamma\frac{d^2v_x}{dt^2} = \frac{q}{c}\frac{dv_y}{dt}B_0 \\ m\gamma\frac{dv_y}{dt} = -\frac{q}{c}v_x B_0 \implies m\gamma\frac{d^2v_y}{dt^2} = -\frac{q}{c}\frac{dv_x}{dt}B_0 \\ m\gamma\frac{dv_z}{dt} = 0 \end{cases} \tag{197}$$

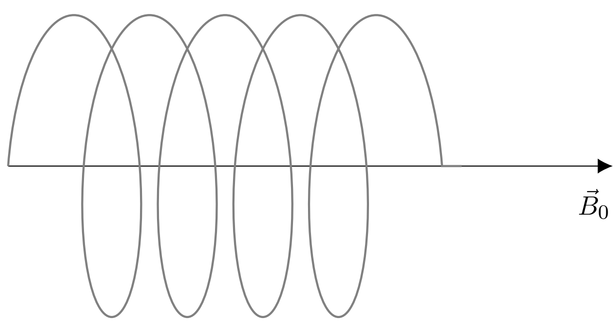
The x and y components change because of circular motion. We obtain

$$\frac{d^2v_x}{dt^2} = -\underbrace{\left(\frac{qB_0}{mc\gamma}\right)^2}_{\Omega} v_x \tag{198}$$

Therefore

$$\begin{cases} v_x(t) = v_{0,x}\cos(\Omega t) \\ v_y(t) = -v_{0,y}\sin(\Omega t) \end{cases} \tag{199}$$

The unperturbed trajectory is what we expect.



Let's define the **pitch angle**

$$\mu = \cos\theta \tag{200}$$

So that we can write

$$v_{0,z} = v_0\cos\theta = v_0\mu \tag{201}$$

II. Motion of a charged particle in a perturbed magnetic field $\vec{B}_0 + \delta\vec{B}$

We are now to consider a more realistic plasma, where in addition to B_0 we also have perturbations

$$\vec{B} = \vec{B}_0 + \delta\vec{B} \quad \text{with} \quad \langle\delta\vec{B}\rangle = 0, \qquad |\delta\vec{B}| \ll \vec{B}_0 \tag{202}$$

We have already found these perturbations in terms of Alfven waves, that move along $\vec{B}_0$. We have also a small electric field $\vec{E}$ in the perpendicular direction.
We will use the reference frame of the Alfven waves. In this case we shall have $\vec{E} = 0$. Remember that $v_A = \frac{B_0}{\sqrt{4\pi\rho}} \sim$ few km per second.
We also assume that the Alfven waves are circularly polarized

$$\delta B_x = \mp i\delta B_y \tag{203}$$

$$\delta B_y = \delta B \exp\left[i\left(kz - \omega t + \phi\right)\right] = \delta B\cos(kz - \omega t + \phi) + i\delta B\sin(kz - \omega t + \phi) \tag{204}$$

where ϕ is a generic phase.

$$\delta B_x = \mp i\delta B_y = \mp i\delta B\cos(kz - \omega t + \phi) \pm \delta B\sin(kz - \omega t + \phi) \tag{205}$$

Since only the real part has a physical meaning we shall write

$$\begin{cases} \delta B_x = \pm\delta B\sin(kz - \omega t + \phi) \\ \delta B_y = \delta B\cos(kz - \omega t + \phi) \end{cases} \tag{206}$$

The Lorentz factor is still constant, since the magnetic field is doing no work.

$$m\gamma\frac{d\vec{v}}{dt} = \frac{q}{c}\left[\vec{v} \times \left(\vec{B}_0 + \delta\vec{B}\right)\right] \tag{207}$$

In terms of components:

$$m\gamma\begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \\ \frac{dv_z}{dt} \end{pmatrix} = \frac{q}{c}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ \delta B_x & \delta B_y & B_0 \end{vmatrix} = \begin{cases} m\gamma\frac{dv_x}{dt} = \frac{q}{c}\left(v_y B_0 - v_z\delta B_y\right) \\ m\gamma\frac{dv_y}{dt} = -\frac{q}{c}\left(v_x B_0 - v_z\delta B_x\right) \\ m\gamma\frac{dv_z}{dt} = \frac{q}{c}\left(v_x\delta B_y - v_y\delta B_x\right) \end{cases} \tag{208}$$

The x and y components are weakly affected: at zero order are unperturbed. The pitch angle will be affected:

$$\begin{cases} v_x(t) = v(1-\mu^2)^{1/2}\cos(\Omega t) \\ v_y(t) = -v(1-\mu^2)^{1/2}\sin(\Omega t) \end{cases} \tag{209}$$

So we shall analyze the z component:

$$\begin{aligned} \frac{dv_z}{dt} &= \frac{q}{c}\frac{\delta B v}{m\gamma}\left((1-\mu^2)^{1/2}\cos(\Omega t)\cos(kz - \omega t + \phi) \pm (1-\mu^2)^{1/2}\sin(\Omega t)\sin(kz - \omega t + \phi)\right) \\ &= \frac{q}{c}\frac{\delta B v}{m\gamma}(1-\mu^2)^{1/2}\cos\left[\Omega t \mp kz \pm \omega t \mp \phi\right] \end{aligned} \tag{210}$$

Since $w = kv_A$ is small because $v \gg v_A$, we can neglect the ωt term, so, using the fact that perturbations are small and $z \sim v\mu t$

$$\frac{dv_z}{dt} = \frac{q\delta B v}{mc\gamma}(1-\mu^2)^{1/2}\cos\left[\Omega t \mp kv\mu t \mp \phi\right] \tag{211}$$

What we have found is how the pitch angle changes

$$\frac{d\mu}{dt} = \frac{q\delta B}{mc\gamma}(1-\mu^2)^{1/2}\cos\left[\Omega t \mp kv\mu t \mp \phi\right] \tag{212}$$

III. Equation of diffusive motion

The pitch angle is fluctuating, even though $\langle\Delta\mu\rangle_T = 0$. What are the conditions to have **diffusive motion**? The trademark is that the square of the distance scales with time. What happens to the square of fluctuations:

$$\begin{aligned}\langle\Delta\mu\Delta\mu\rangle_{T,\phi} &= \frac{q^2\delta B^2}{(mc\gamma)^2}\frac{1-\mu^2}{2\pi}\int d\phi\int_0^T dt\int_0^T dt'\cos(\Omega t\mp kv\mu t\mp\phi)\cos(\Omega t'\mp kv\mu t'\mp\phi)\\ &= \left(\frac{q\delta B}{mc\gamma}\right)^2(1-\mu^2)\frac{1}{2}\int_0^T dt\int_0^T dt'\cos\left[(\Omega\mp v\mu k)(t-t')\right]\\ &= \left(\frac{q\delta B}{mc\gamma}\right)^2(1-\mu^2)\frac{2\pi}{2}T\delta(\Omega\mp vk\mu)\end{aligned}\tag{213}$$

where we have recognized a representation of the δ-function on the last passage[8]. So, in the limit of large T:

$$\langle\Delta\mu\Delta\mu\rangle = \left(\frac{q\delta B}{mc\gamma}\right)^2(1-\mu^2)\pi T\delta(\Omega\mp vk\mu)\tag{214}$$

Since we have a δ-function it should be $\Omega = vk\mu$

$$\Delta\mu\sim\sqrt{t}\tag{215}$$

The particle is developing diffusive motion. If $\mu = 1$, the particle is moving along $\vec{B}_0$. Then we have

$$k\sim\frac{\Omega}{v\mu} = \frac{1}{r_L\mu}\tag{216}$$

where $r_L = \frac{v}{\Omega}$ is the **Larmor radius**. The motion becomes diffusive in the pitch angle if the wave number is $\frac{1}{r_L\mu}$. When it is otherwise the particle is moving ballistically and there is no **resonance condition**.
So we have found

$$\langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle = \left(\frac{q\delta B}{mc\gamma}\right)^2(1-\mu^2)\frac{\pi}{\mu v}\delta\left[k\mp\frac{\Omega}{v\mu}\right]\tag{217}$$

The **diffusion coefficient** in pitch angle is defined as

$$D_{\mu\mu} = \frac{1}{2}\langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle\tag{218}$$

It says how long it takes to change the pitch angle.
Let's rewrite it in another way

$$\begin{aligned}\langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle &= \left(\frac{qB_0}{mc\gamma}\right)^2\left(\frac{\delta B}{B_0}\right)^2(1-\mu^2)\frac{\pi}{\mu v}\delta\left[k\mp\frac{\Omega}{v\mu}\right]\\ &= \Omega^2\left(\frac{\delta B}{B_0}\right)^2(1-\mu^2)\frac{\pi}{\mu v}\delta\left[k\mp\frac{\Omega}{v\mu}\right]\end{aligned}\tag{219}$$

The term $\left(\frac{\delta B}{B_0}\right)^2$ gives us the strength of the fluctuations. But we shall take into consideration that in a plasma we have many waves with different wave numbers k, so it is a good idea to integrate over k. Therefore

$$\begin{aligned}D_{\mu\mu} &= \Omega^2(1-\mu^2)\frac{\pi}{v\mu}\int dk\frac{\delta B^2(k)}{B_0^2}\delta\left[k\mp\frac{\Omega}{v\mu}\right]\\ &= \Omega^2(1-\mu^2)\frac{\pi}{v\mu}\frac{\delta B^2(k_{\rm res})}{B_0^2}\frac{\Omega}{\Omega}\\ &= \pi\Omega(1-\mu^2)k_{\rm res}F(k_{\rm res})\end{aligned}\tag{220}$$

[8] We have also used the prosthaphaeresis formula for the product:

$$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$$

Note that the term $\cos(\alpha+\beta)$ would average to zero, so only the second term can contribute where we have defined

$$k_{\mathrm{res}} = \pm\frac{\Omega}{v\mu} \tag{221}$$

and

$$\mathcal{F}(k_{\mathrm{res}}) = \frac{\delta B^2(k_{\mathrm{res}})}{B_0^2} \tag{222}$$

Since $k_{\mathrm{res}}\mathcal{F}(k_{\mathrm{res}})$ is dimensionless, and Ω is the inverse of time

$$D_{\mu\mu} = \frac{1}{[T]} \tag{223}$$

Let us now define the diffusion coefficient in terms of the angle θ:

$$D_{\theta\theta} = \frac{1}{2}\langle\frac{\Delta\theta\Delta\theta}{\Delta t}\rangle = \frac{1}{1-\mu^2}D_{\mu\mu} \tag{224}$$

$$\implies D_{\theta\theta} = \pi\Omega k_{\mathrm{res}}F(k_{\mathrm{res}}) \tag{225}$$

So we can calculate the amount of time it takes to change θ by 1rad

$$\tau_{\mathrm{diff}} \sim \frac{1}{D_{\theta\theta}} \sim \frac{1}{\Omega\pi}\frac{1}{k_{\mathrm{res}}F(k_{\mathrm{res}})} \gg \frac{1}{\Omega} \tag{226}$$

Remember that Ω is the giration time. The particle has moved from ballistic to diffusive motion. The particle can even go backwards. The effect is dramatic! In one diffusion time we can have

$$\Delta z = v\tau_{\mathrm{diff}} \tag{227}$$

$$D_{zz} = \frac{1}{2}\langle\frac{\Delta z\Delta z}{\Delta t}\rangle = v^2\tau_{\mathrm{diff}} = \frac{v^2}{\pi\Omega k_{\mathrm{res}}F(k_{\mathrm{res}})} \tag{228}$$

$$\implies D_{zz} \sim \underbrace{\frac{1}{3}r_L(p)v}_{\text{Bohm diff. coeff.}}\;\frac{1}{k_{\mathrm{res}}F(k_{\mathrm{res}})} \tag{229}$$

where we approximated $\pi \sim 3$. We have obtained the expression for D_{zz} in quasi-linear theory.

$$[D_{zz}] = \frac{L^2}{T} \tag{230}$$

So a time scale will be given by

$$\tau_{esc} = \frac{L^2}{D_{zz}} \tag{231}$$

This is the time it takes to leave a region of size L. Let's put now some numbers. What is r_L for 1GeV electron?

$$r_L = \frac{pc}{eB_0} = 3\times 10^{12}\mathrm{cm}\ (\mathrm{E_{Gev}}) \tag{232}$$

where we have assumed $B_0 \sim 1\mu$G. The number we obtained is small compared to the typical galactic sizes (∼kpc $\sim 10^{21-22}$cm). So

$$D(p) = \frac{1}{3}\left(3\times 10^{12}\mathrm{cm\ E_{Gev}}\right)\left(3\times 10^{10}\right)\frac{1}{k_{\mathrm{res}}F(k_{\mathrm{res}})} \tag{233}$$

On the other hand, from observations we know that the time required to leave our galaxy is of order of:

$$\frac{H^2}{D} = 100\mathrm{Myr} = 3\times 10^{15}\mathrm{s} \tag{234}$$

Using $H \sim 3\text{kpc}$[9], we can calculate D_{zz} and therefore estimate the order of magnitude of the perturbations:

$$k_{\text{res}} F(k_{\text{res}}) = \left(\frac{10^{43}}{3 \times 10^{22}} \right)^{-1} \cdot 3 \times 10^{15} \sim 10^{-5} \tag{235}$$

These are really small fluctuations, but they have gigantic consequences on the transport of charge particles.

Some Comments

- In our calculations so far we have put ourselves in the reference system of Alfven waves: therefore the electric field $\vec{E}$ could be considered negligible
- We have found an important consequence, $\Delta\mu\Delta\mu \sim t$, which is a distinctive feature of diffusive motion.
- We have resonance with Alfven waves when $k \sim 1/r_L$, depending on μ we pick a different value for k
- When $\mu \sim 0$, k becomes very large
- At 1GeV, $r_L = 10^{12}$cm, which is a gigantic value compared to dissipative scales
- At 10^6GeV (1PeV), r_L ∼pc: it may become concerning find wavenumbers to resonate with
- At 10^{20}GeV (ultra high energy CR), r_L is bigger than the size of a galaxy. These particles do not diffuse, but move in a ballistical way

We have found that the motion of a charged particle in a magnetic field that has an ordered component B_0 plus some perturbations, Alfven waves, can change its direction of motion erratically as a consequence of resonances with Alfven waves. This leads to the phenomenon of pitch angle diffusion, which in turn, as we'll see, leads to spatial diffusion.

iii.1 Equation of diffusive motion in pitch angle

Let now us tackle the problem of what is the distribution of these diffusive particles. We can think of the following gedanken experiment: shoot a beam of particles in the same direction. They enter a plasma with a magnetic field $\vec{B}$ and Alfven waves. We expect at least an isotropic distribution. We will follow the Focker-Planck approach, developed by London in $'20 -' 30$s.
It is useful to introduce a distribution function

$$f(\vec{x}, \vec{p}, t) \to f(z, \mu, t) \tag{236}$$

since the motion is in the z direction, Alfven waves are stationary (so that p does not change), and we assume azimutal symmetry (so that μ is the only angle that maters).
What happens to f when we move of a time Δt?

$$f(z + v\mu\Delta t, \mu, t + \Delta t) \tag{237}$$

We introduce another function, $\Psi(\mu, \Delta\mu)$, representing the probability that the particle changes its angle by an amount $\Delta\mu$. Being a probability

$$\int \Psi(\mu, \Delta\mu) d\Delta\mu = 1 \tag{238}$$

Therefore we can write:

$$f(z + v\mu\Delta t, \mu, t + \Delta t) = \int d\Delta\mu f(z, \mu - \Delta\mu, t)\Psi(\mu - \Delta\mu, \Delta\mu) \tag{239}$$

[9]Remember that

$$1\text{pc} = 3.086 \times 10^{18}\text{cm}$$

We now apply perturbation theory and Taylor expand. For the LHS:

$$f(z+v\mu\Delta t,\mu,t+\Delta t)=f+\frac{\partial f}{\partial z}v\mu\Delta t+\frac{\partial f}{\partial t}\Delta t \tag{240}$$

We can do the same for the RHS, but this time till the second order

$$f(z,\mu-\Delta\mu,t)=f-\frac{\partial f}{\partial \mu}\Delta\mu+\frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\Delta\mu\Delta\mu \tag{241}$$

and

$$\Psi(\mu-\Delta\mu,\Delta\mu)=\Psi-\frac{\partial \Psi}{\partial \mu}\Delta\mu+\frac{1}{2}\frac{\partial^2 \Psi}{\partial \mu^2}\Delta\mu\Delta\mu \tag{242}$$

We can now insert all these quantities in (239) and do the multiplications remembering to retain only terms up to second order

$$\begin{aligned}
f+\frac{\partial f}{\partial z}v\mu\Delta t+\frac{\partial f}{\partial t}\Delta t &= \int d\Delta\mu\left[f-\frac{\partial f}{\partial \mu}\Delta\mu+\frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\Delta\mu\Delta\mu\right]\left[\Psi-\frac{\partial \Psi}{\partial \mu}\Delta\mu+\frac{1}{2}\frac{\partial^2 \Psi}{\partial \mu^2}\Delta\mu\Delta\mu\right]\\
\cancel{f}+\frac{\partial f}{\partial z}v\mu\Delta t+\frac{\partial f}{\partial t}\Delta t &= \cancel{f\underbrace{\int d\Delta\mu\Psi}_{1}}-\int d\Delta\mu f\frac{\partial \Psi}{\partial \mu}\Delta\mu+\frac{1}{2}\int d\Delta\mu f\frac{\partial^2 \Psi}{\partial \mu^2}\Delta\mu\Delta\mu\\
&\quad-\int d\Delta\mu\frac{\partial f}{\partial \mu}\Delta\mu\Psi+\int d\Delta\mu\frac{\partial f}{\partial \mu}\frac{\partial \Psi}{\partial \mu}\Delta\mu\Delta\mu\\
&\quad+\frac{1}{2}\int d\Delta\mu\frac{\partial^2 f}{\partial \mu^2}\Psi\Delta\mu\Delta\mu\\
\frac{\partial f}{\partial z}v\mu\Delta t+\frac{\partial f}{\partial t}\Delta t &= -\int d\Delta\mu\Delta\mu\frac{\partial}{\partial \mu}\left(\Psi f\right)+\int d\Delta\mu\Delta\mu\Delta\mu\left[\frac{1}{2}\frac{\partial^2 \Psi}{\partial \mu^2}f+\frac{\partial f}{\partial \mu}\frac{\partial \Psi}{\partial \mu}+\frac{1}{2}\Psi\frac{\partial^2 f}{\partial \mu^2}\right]\\
\frac{\partial f}{\partial z}v\mu\Delta t+\frac{\partial f}{\partial t}\Delta t &= -\int d\Delta\mu\Delta\mu\frac{\partial}{\partial \mu}\left(\Psi f\right)+\int d\Delta\mu\Delta\mu\Delta\mu\frac{1}{2}\frac{\partial^2}{\partial \mu^2}\left(\Psi f\right)\\
\frac{\partial f}{\partial z}v\mu\Delta t+\frac{\partial f}{\partial t}\Delta t &\equiv -\frac{\partial}{\partial \mu}\left(Af\right)+\frac{1}{2}\frac{\partial^2}{\partial \mu^2}\left(Bf\right)
\end{aligned} \tag{243}$$

Where we have recognized that

$$\begin{aligned}
\frac{\partial^2}{\partial \mu^2}\left(\Psi f\right) &= \frac{\partial}{\partial \mu}\left[\frac{\partial f}{\partial \mu}\Psi+f\frac{\partial \Psi}{\partial \mu}\right]\\
&= \frac{\partial^2 f}{\partial \mu^2}\Psi+\frac{\partial f}{\partial \mu}\frac{\partial \Psi}{\partial \mu}+\frac{\partial f}{\partial \mu}\frac{\partial \Psi}{\partial \mu}+f\frac{\partial^2 \Psi}{\partial \mu^2}\\
&= \frac{\partial^2 f}{\partial \mu^2}\Psi+2\frac{\partial f}{\partial \mu}\frac{\partial \Psi}{\partial \mu}+f\frac{\partial^2 \Psi}{\partial \mu^2}
\end{aligned} \tag{244}$$

and where we have defined the following quantities:

$$A(\mu)=\int d\Delta\mu\Delta\mu\Psi; \qquad B(\mu)=\int d\Delta\mu\Delta\mu\Delta\mu\Psi \tag{245}$$

Note that in the last passage of (243) we are keeping into account that f does not depend on μ.
We now use the property of **detailed balance** for the probability distribution Ψ:

$$\Psi(\mu,-\Delta\mu)=\Psi(\mu-\Delta\mu,\Delta\mu) \tag{246}$$

which we can Taylor expand as well

$$\cancel{\Psi(\mu,-\Delta\mu)}=\Psi(\mu-\Delta\mu,\Delta\mu)=\cancel{\Psi(\mu,\Delta\mu)}-\frac{\partial \Psi}{\partial \mu}\Delta\mu+\frac{1}{2}\frac{\partial^2 \Psi}{\partial \mu^2}\Delta\mu\Delta\mu \tag{247}$$

If we integrate (247) in $d\Delta\mu$, we obtain

$$\begin{aligned} -\frac{\partial}{\partial\mu}\underbrace{\int d\Delta\mu\Psi\Delta\mu}_{A}+\frac{1}{2}\frac{\partial^2}{\partial\mu^2}\underbrace{\int d\Delta\mu\Delta\mu\Delta\mu\Psi}_{B} &= 0 \\ -\frac{\partial A}{\partial\mu}+\frac{1}{2}\frac{\partial^2 B}{\partial\mu^2} &= 0 \\ \frac{\partial}{\partial\mu}\left[A-\frac{1}{2}\frac{\partial B}{\partial\mu}\right] &= 0 \end{aligned} \tag{248}$$

So the quantity $A-\frac{1}{2}\frac{\partial B}{\partial\mu}$ is constant in μ. Let's go into details:

$$\frac{\partial B}{\partial\mu}=\frac{\partial}{\partial\mu}\int d\Delta\mu\Delta\mu\Delta\mu\Psi(\mu,\Delta\mu) \tag{249}$$

and use again the property of detailed balance

$$\frac{\partial\Psi}{\partial\mu}(\mu,\Delta\mu)=\frac{\partial\Psi}{\partial\mu}(\mu-\Delta\mu,\Delta\mu)=\frac{\partial\Psi}{\partial\mu}(\mu+\Delta\mu,-\Delta\mu) \tag{250}$$

Introduce a new variable $\mu'=\mu+\Delta\mu$, so that $\Delta\mu=\mu'-\mu$ and

$$\frac{\partial\Psi}{\partial\mu}(\mu,\Delta\mu)=\frac{\partial\Psi}{\partial\mu}(\mu',\mu-\mu')=-\frac{\partial}{\partial\Delta\mu}\Psi(\mu+\Delta\mu,-\Delta\mu) \tag{251}$$

We now replace this quantity in the definition of $\frac{\partial B}{\partial\mu}$:

$$\begin{aligned} \frac{\partial B}{\partial\mu} &= -\int d\Delta\mu\Delta\mu\Delta\mu\frac{\partial}{\partial\Delta\mu}\Psi(\mu+\Delta\mu,-\Delta\mu)\overset{IBP}{=} \\ &= \int d\Delta\mu\Psi(\mu+\Delta\mu,-\Delta\mu)\frac{\partial}{\partial\mu}(\Delta\mu\Delta\mu)\overset{\text{det. bal.}}{=} \\ &= \int d\Delta\mu\Psi(\mu,\Delta\mu)2\Delta\mu \\ &= 2A \end{aligned} \tag{252}$$

Let's go back to

$$\begin{aligned} \frac{\partial f}{\partial t}\Delta t+v\mu\frac{\partial f}{\partial z}\Delta t &= -\frac{\partial}{\partial\mu}\left[Af-\frac{1}{2}\frac{\partial}{\partial\mu}(fB)\right] \\ &= -\frac{\partial}{\partial\mu}(Af)+\frac{1}{2}\frac{\partial}{\partial\mu}\left[\frac{\partial B}{\partial\mu}f+B\frac{\partial f}{\partial\mu}\right] \\ &= -\cancel{\frac{\partial}{\partial\mu}(Af)}+\cancel{\frac{\partial}{\partial\mu}(Af)}+\frac{1}{2}\frac{\partial}{\partial\mu}\left[B\frac{\partial f}{\partial\mu}\right] \\ &= \Delta t\frac{\partial}{\partial\mu}\left[D_{\mu\mu}\frac{\partial f}{\partial\mu}\right] \end{aligned} \tag{253}$$

where the last RHS is the coefficient diffusion in pitch angle and represents a **collision term**

$$D_{\mu\mu}=\frac{1}{2}\langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle=\frac{1}{2}\int d\Delta\mu\frac{\Delta\mu\Delta\mu}{\Delta t}\Psi \tag{254}$$

We have found what happens to the distribution f as a consequence of the diffusion in pitch angle. The distribution opens up.

$$\frac{\partial f}{\partial t}+v\mu\frac{\partial f}{\partial z}=\frac{\partial}{\partial\mu}\left[D_{\mu\mu}\frac{\partial f}{\partial\mu}\right] \tag{255}$$

The LHS of (255) is the total time derivative of the distribution and also the first part of the Valsov equation. The force term then is a collision term now. Therefore we can say that under the effect of Alfven waves sitting in the plasma, the distribution of particles follows (255).

iii.2 Equation of diffusive motion in space

We need to know the properties of diffusion in space, not in pitch angle. If f is weakly dependent on μ, after waiting long enough the distribution will become isotropized. On the other hand, if the system is relativistic and very dependent on μ, then it becomes impossible to isotropize the distribution.
Let's study the case when f is weakly dependent on μ. We want to go from the distribution in pitch angle to the distribution in spatial diffusion. We have

$$\frac{\partial f}{\partial t} + v\mu\frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right] \tag{256}$$

We apply to all terms the operator $\frac{1}{2}\int_{-1}^{1} d\mu$ and define also

$$M = \frac{1}{2}\int_{-1}^{1} d\mu f(\mu) \tag{257}$$

We note that in case f is constant in μ then $M = f$. Therefore the operator M identifies the isotropic part of f.

$$\frac{\partial M}{\partial t} + \frac{v}{2}\frac{\partial}{\partial z}\int_{-1}^{1} d\mu\mu f = \underbrace{D_{\mu\mu}\frac{\partial f}{\partial \mu}}_{0} \tag{258}$$

since $D_{\mu\mu} \sim (1-\mu^2)$.
Let us call

$$\frac{1}{2}\int_{-1}^{1} d\mu v\mu f = J \tag{259}$$

It is indeed a current! So we obtain

$$\frac{\partial M}{\partial t} + \frac{\partial J}{\partial z} = 0 \tag{260}$$

M evolves in time only if there is a current. Let's concentrate on the current:

$$\begin{aligned} J &= \frac{1}{2}v\int_{-1}^{1} d\mu f\mu = -\frac{v}{4}\int_{-1}^{1} d\mu f\frac{\partial}{\partial \mu}(1-\mu^2) \\ &= \underbrace{\left[-\frac{v}{4}(1-\mu^2)f\right]_{-1}^{+1}}_{0} + \frac{v}{4}\int_{-1}^{+1} d\mu(1-\mu^2)\frac{\partial f}{\partial \mu} \end{aligned} \tag{261}$$

where we used the tautology $\mu = -\frac{1}{2}\frac{\partial}{\partial \mu}(1-\mu^2)$
Now let's go back to the initial equation (256) and integrate between -1 and μ

$$\frac{\partial}{\partial t}\int_{-1}^{\mu} d\mu' f + \int_{-1}^{\mu} d\mu' v\mu'\frac{\partial f}{\partial z} = D_{\mu\mu}\frac{\partial f}{\partial \mu} \tag{262}$$

Multiply all terms by $\frac{1-\mu^2}{D_{\mu\mu}}$. Moreover, in all quantities where μ does not appear explicitly we assume f isotropic.

$$(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial M}{\partial t}(1+\mu) + v\frac{\partial M}{\partial z}\frac{1}{2}(\mu^2-1)\frac{(1-\mu^2)}{D_{\mu\mu}} \tag{263}$$

Now integrate between -1 and $+1$:

$$\int_{-1}^{+1} d\mu(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{\partial M}{\partial t}\int_{-1}^{+1} d\mu\frac{(1-\mu^2)}{D_{\mu\mu}}(1+\mu) - \frac{1}{2}v\frac{\partial M}{\partial z}\int_{-1}^{+1} d\mu\frac{(1-\mu^2)^2}{D_{\mu\mu}} \tag{264}$$

Remembering that $J = \frac{v}{4}\int_{-1}^{+1} d\mu(1-\mu^2)\frac{\partial f}{\partial \mu}$, we obtain

$$J = \frac{v}{4}\frac{\partial M}{\partial t}\int_{-1}^{+1} d\mu \frac{(1-\mu^2)}{D_{\mu\mu}}(1+\mu) - \frac{v^2}{8}\frac{\partial M}{\partial z}\int_{-1}^{+1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}} \tag{265}$$

Now we define the following two quantities:

$$K_t = \frac{v}{4}\int_{-1}^{+1} d\mu \frac{(1-\mu^2)(1+\mu)}{D_{\mu\mu}}; \qquad K_z = \frac{v^2}{8}\int_{-1}^{+1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}} \tag{266}$$

We'll explore later the physical meaning of these two quantities. Let's exploit now the earlier found relation $\frac{\partial M}{\partial t} + \frac{\partial J}{\partial z} = 0$ and write

$$\begin{aligned} J &= K_t \frac{\partial M}{\partial t} - K_z \frac{\partial M}{\partial z} \\ &= -K_t \frac{\partial J}{\partial z} - K_z \frac{\partial M}{\partial z} \end{aligned} \tag{267}$$

Therefore

$$\frac{\partial M}{\partial t} = -\frac{\partial}{\partial z}\left[-K_t \frac{\partial J}{\partial z} - K_z \frac{\partial M}{\partial z}\right] \tag{268}$$

As order of magnitude:

$$K_t \frac{\partial J}{\partial z} \sim \frac{v}{D_{\mu\mu}}\frac{J}{z} \tag{269}$$

$$K_z \frac{\partial M}{\partial z} \sim \frac{v^2}{D_{\mu\mu}}\frac{M}{z} \tag{270}$$

So

$$J = \frac{v}{2}\int_{-1}^{+1} d\mu\mu\left[M + \epsilon\mu\right] \sim \frac{1}{2}v\epsilon\frac{2}{3} \sim v\epsilon \ll vM \tag{271}$$

where we have neglected the M term since it does not contribute to the current.

$$\frac{\partial M}{\partial t} \sim \frac{\partial}{\partial z}\left[K_z \frac{\partial M}{\partial z}\right] \tag{272}$$

This is the **diffusion equation in space** and K_z is the diffusion coefficient in space.

iii.3 Equation of diffusive motion with moving plasma

Note that we have assumed the background to be at rest. This is the reason why we do not have any terms that indicate motion of plasma in our equation.
We have also assumed that the waves were stationary, so that we don't have an electric field.
If we weren't in the reference frame of Alfven waves, we would have had nonzero perturbations in the electric field.
What would have been the change in particle's momentum then? We have a particle of momentum p that scatters elastically with a wave. The change in momentum will be of order $\Delta p \sim m v_A \gamma \Delta\mu$, therefore

$$\langle\frac{\Delta p \Delta p}{\Delta t}\rangle \sim (m v_A \gamma)^2 \langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle = (m\gamma v_A)^2 \Omega k_{\rm res} F(k_{\rm res}) = (m\gamma v_A)^2 \frac{c^2}{D_{zz}} = (m\gamma c)^2 \frac{v_A^2}{c^2}\frac{1}{\tau_{\rm diff}} \tag{273}$$

where in the last passage we used that $D_{zz} \sim v^2 \tau_{\rm diff}$ and since particles are considerate to be relativistic $v \sim c$.
So we have obtained that:

$$D_{pp} = \langle\frac{\Delta p \Delta p}{\Delta t}\rangle = p^2 \left(\frac{v_A}{c}\right)^2 \frac{1}{\tau_{\rm diff}} \tag{274}$$

where we have defined τ_{diff} as the time necessary for a deflection of 1rad.

$$\tau_p = \frac{p^2}{D_{pp}} = \left(\frac{c}{v_A}\right)^2 \tau_{\text{diff}} \gg \tau_{\text{diff}} \tag{275}$$

The particle momentum cannot possibly change by the time the particle diffuses. As long as $v_A \ll c$ (non relativistic situation!), this reasoning holds. In case $v_A \sim c$, we have that this acceleration method is non negligible, as in the phenomena of gamma ray bursts. So in our non relativistic situation it is a good idea to take the reference frame of Alfven waves.

Observe that the diffusion equation we have found is equivalent to the **heat equation**.

Our next step will be that of generalizing equation $\frac{\partial M}{\partial t} = \frac{\partial}{\partial z}\left[K_z \frac{\partial M}{\partial z}\right]$ to the case of moving plasma. We have also to account for the change in the momentum of the particle due to Lorenzt transformations.
This will finally give us the equation to describe the transport of cosmic rays in galaxy and, in general, all non thermal phenomena.

So we started with the Vlasov equation and obtained $\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu} \frac{\partial f}{\partial \mu}\right]$. The distribution function f is nearly isotropic, so that is why it makes sense to introduce the $M(z,p,t) = \frac{1}{2}\int_{-1}^{+1} d\mu f$ quantity. We have also found that M satisfies $\frac{\partial M}{\partial t} = \frac{\partial}{\partial z}\left[K_z \frac{\partial M}{\partial z}\right]$.

We want now to analyze a moving plasma, characterized by velocity gradients (so not a uniform velocity). We call

$$u \equiv \text{plasma velocity}, \qquad u \ll c \tag{276}$$

So now the velocity of particles in the plasma for an external observer will be $u + v\mu$. Since, in principle, $u \equiv u(z)$ we cannot just substitute $v\mu$ with $u + v\mu$.
Let's revise the equations that describe the change of reference frame:

$$\begin{pmatrix} \frac{E'}{c} \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_z \end{pmatrix} \tag{277}$$

Remember also that

$$\begin{cases} p'_x = p_x \\ p'_y = p_y \end{cases} \tag{278}$$

Here

$$\beta = \frac{u(z)}{c}, \text{ and } \gamma \sim 1 \tag{279}$$

if the system is not relativistic. Therefore

$$\begin{cases} \frac{E'}{c} = \gamma \frac{E}{c} - \beta\gamma p\mu \\ p'_z = -\beta\gamma \frac{E}{c} + \gamma p\mu \end{cases} \tag{280}$$

Now:

$$\frac{\partial f}{\partial z} \to \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p'_z}\frac{dp'_z}{dz} = \frac{\partial f}{\partial z} - \frac{\partial f}{\partial p'_z}\frac{1}{c^2}\frac{du(z)}{dz}E \tag{281}$$

Let's go back to the expression

$$\begin{aligned} (v\mu + u)\frac{\partial f}{\partial z} \to & (v\mu + u)\left[\frac{\partial f}{\partial z} - \frac{E}{c^2}\frac{du}{dz}\frac{\partial f}{\partial p'_z}\right] \\ &= (v\mu + u)\frac{\partial f}{\partial z} - \frac{E}{c^2} v\mu \frac{du}{dz}\frac{\partial f}{\partial p'_z} \end{aligned} \tag{282}$$

where we have considered negligible the last term. We now want to write the term $\frac{\partial f}{\partial p'_z}$ in terms of known quantities. We have that $p'_z = p_\parallel$ and we drop the primed indexes. We'll use

$$\begin{cases} p_\parallel = p\mu \\ p_\perp = p(1-\mu^2)^{1/2} \end{cases} \implies \begin{cases} p = \sqrt{p_\parallel^2 + p_\perp^2} \\ \mu = \frac{p_\parallel}{\sqrt{p_\parallel^2+p_\perp^2}} \end{cases} \tag{283}$$

Therefore:

$$\begin{aligned} \frac{\partial f}{\partial p'_z} &= \frac{\partial f}{\partial p}\frac{\partial p}{\partial p_\parallel} + \frac{\partial f}{\partial \mu}\frac{\partial \mu}{\partial p_\parallel} \\ &= \frac{\partial f}{\partial p}\mu + \frac{\partial f}{\partial \mu}\frac{1-\mu^2}{p} \end{aligned} \tag{284}$$

and

$$\frac{\partial f}{\partial t} + (v\mu + u)\frac{\partial f}{\partial z} - \frac{E}{c^2}\frac{du}{dz}v\mu\left[\frac{\partial f}{\partial p}\mu + \underbrace{\frac{1-\mu^2}{p}\frac{\partial f}{\partial \mu}}_{\text{negligible}}\right] = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right] \tag{285}$$

We how now obtained proportionality to $\frac{du}{dz}$. Note that the negligible term is such because it is proportional to the anisotropy.
Now we apply the M operator and obtain

$$\begin{aligned} \frac{\partial M}{\partial t} + u\frac{\partial M}{\partial z} - \frac{E}{c^2}\frac{du}{dz}\frac{1}{2}\int_{-1}^{+1} d\mu v\mu^2\frac{\partial f}{\partial p} &= \frac{\partial}{\partial z}\left[K_z\frac{\partial M}{\partial z}\right] \\ \frac{\partial M}{\partial t} + u\frac{\partial M}{\partial z} - \frac{E}{c^2}\frac{du}{dz}v\frac{\partial M}{\partial p}\frac{1}{2}\int_{-1}^{+1} d\mu\mu^2 &= \frac{\partial}{\partial z}\left[K_z\frac{\partial M}{\partial z}\right] \\ \frac{\partial M}{\partial t} + u\frac{\partial M}{\partial z} - \frac{1}{3}\frac{Ev}{c^2}\frac{du}{dz}\frac{\partial M}{\partial p} &= \frac{\partial}{\partial z}\left[K_z\frac{\partial M}{\partial z}\right] \end{aligned} \tag{286}$$

Observe that

$$\frac{E}{c^2}v = \frac{m\cancel{c^2}\gamma}{\cancel{c^2}}v = m\gamma v = p \tag{287}$$

Putting everything together we finally at the equation describing the transport of non thermal particles under the diffusion motion of the pitch angle:

$$\frac{\partial M}{\partial t} + u\frac{\partial M}{\partial z} - \frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial M}{\partial p} = \frac{\partial}{\partial z}\left[K_z\frac{\partial M}{\partial z}\right] \tag{288}$$

where $u\frac{\partial M}{\partial z}$ is the **advection/convection** term, $-\frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial M}{\partial p}$ represents the **gradient** term and $\frac{\partial}{\partial z}\left[K_z\frac{\partial M}{\partial z}\right]$ is the **spatial diffusion** term.
An interesting observation is that we cannot eliminate the gradients, even if we try to change reference frame. This fact will lead to a lot of interesting physics.

Diffusive motion equations

We have found that in presence of magnetic field perturbations (Alfven waves) particles experience **diffusive motion**, where the diffusion coefficient in pitch angle is defined as:

$$D_{\mu\mu} = \frac{1}{2}\langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle$$

In an analogous way one can define the diffusion coefficient in θ angle $D_{\theta\theta} = \frac{1}{2}\langle\frac{\Delta\theta\Delta\theta}{\Delta t}\rangle$ and in space $D_{zz} = \frac{1}{2}\langle\frac{\Delta z\Delta z}{\Delta t}\rangle$. The equation of motion is:

- In pitch angle:
$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right]$$
- In space in the plasma rest frame:
$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial z}\left[K_z\frac{\partial M}{\partial z}\right]$$
- In space with moving plasma (u = plasma velocity):
$$\frac{\partial M}{\partial t} + u\frac{\partial M}{\partial z} - \frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial M}{\partial p} = \frac{\partial}{\partial z}\left[K_z\frac{\partial M}{\partial z}\right]$$

The **transport equation** we have found under the action of **advection** and **diffusion** is very general. It also includes the possibility that the plasma in the background has **gradients of velocity**.
This equation will be be of extreme importance as we can apply it to describe the propagation of charged particles close to a shock front. Here indeed we'll need the velocity gradient description. Moreover, we can apply this equation also to describe what happens to particles after they leave the accelerator and propagate in the Galaxy, reaching eventually the Earth or leaving the Galaxy.

III. Diffusive shock acceleration

The question we are trying to answer is how Nature accelerates particles? We have only two ingredients (remember that only an electric field can change the momentum modulus):

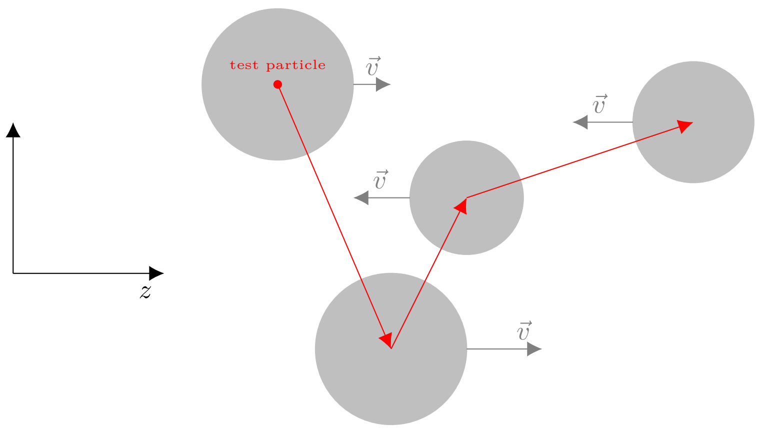
- *Regular acceleration*: we have that $\langle \vec{E} \rangle \neq 0$, so we have to violate the conditions of ideal MHD, which is very difficult
- *Stochastic acceleration*: in this case we respect the condition $\langle \vec{E} \rangle = 0$, but we have that $\langle \vec{E}^2 \rangle \neq 0$. This is the so called **second order Fermi acceleration**

Historical curiosity

Fermi participated to a talk by Alfven in 1947. At those times little was known about the interstellar medium, although cosmic rays have been discovered in 1912 and particles with energies up to 10^{13}GeV have been observed. Fermi came up with the idea of second order acceleration on New Year's Eve night!

I. Fermi second order acceleration

Imagine a bunch of clouds of plasma moving in random directions. A test particle with initial energy E encounters one of this clouds, whose β factor is $\beta = \frac{u}{c}$.



An observer on the cloud will see

$$\begin{cases} E' = \gamma E + \beta\gamma p\mu \\ p'_z = \beta\gamma E + \gamma p\mu \end{cases} \tag{289}$$

The particle gets reflected and its energy for an external observer becomes:

$$E'' = \gamma E' + \beta\gamma p'_z = \gamma^2 E \left[1 + \beta^2 + 2\beta\mu\frac{p}{E}\right] \tag{290}$$

where $\frac{p}{E} \sim v$. Notice that we haven't made any particular assumption: particle and clouds are moving in random directions.

How much the energy has changed with respect to the initial energy E (remember that particles are relativistic, the clouds not)

$$\frac{\Delta E}{E} = \frac{E'' - E}{E} \simeq 2\beta^2 + 2\beta v\mu \tag{291}$$

This result contains a lot of information. First of all note that μ can be anything between -1 and $+1$. Therefore there are values of μ for which energy is increasing and others for which energy is decreasing (imagine a photon impacting on a moving mirror). The configurations, though, are not equally probable. So we ask what is the probability of having an interaction with a pitch angle μ? It will primarily depend on the relative velocity between the particle and the cloud

$$P(\mu) = Av_{\text{rel}} = A\frac{\beta\mu + v}{1 + v\beta\mu} \overset{v\to 1,\beta\ll 1}{\longrightarrow} \sim A(1 + \beta\mu) \tag{292}$$

where A is a constant we can determine imposing that the total probability should be equal to unity.

$$A\int_{-1}^{+1} d\mu(1 + \beta\mu) = 1 \implies A = \frac{1}{2} \tag{293}$$

So the average change in energy can be written as

$$\begin{aligned}\langle\frac{\Delta E}{E}\rangle &= \int_{-1}^{+1} d\mu P(\mu)(2\beta^2 + 2\beta\mu) \\ &= \frac{1}{2}\int_{-1}^{+1} d\mu(1 + \beta\mu)(2\beta^2 + 2\beta\mu) \\ &= \frac{8}{3}\beta^2\end{aligned} \tag{294}$$

Since the energy variation[10] is positive we can conclude that **head-on collisions** are more important than **tail-on collisions**.
We have that $\beta = u/c$, where $u \sim v_A = \frac{B_0}{\sqrt{4\pi\rho}} \sim 10\text{km/s}$. Therefore $\frac{\Delta E}{E} \sim 10^{-8}$, which is very small (this is a second order mechanism). *It is extremely inefficient!* This mechanism is second order because the particle energy is allowed to both increase and decrease depending on the relative velocity between the particle and the cloud.

II. Diffusive shock acceleration

It was not until 1982 that somebody understood how it was possible to have a mechanism first order in β. Let's restate here the conservation equation for our plasma.
We have energy conservation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \tag{295}$$

and momentum conservation[11]

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial z} \tag{296}$$

Now introduce the entropy per unit mass s, satisfying the adiabatic condition

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial z} = 0 \tag{297}$$

Entropy per unit volume would be ρs, so that we can write

$$\frac{\partial(\rho s)}{\partial t} + \vec{\nabla}(\rho s\vec{u}) = 0 \underset{1d}{\implies} \frac{\partial(\rho s)}{\partial t} + \frac{\partial}{\partial z}(\rho s u) = 0 \tag{298}$$

The **enthalpy** of the system is

$$W = E + PV \tag{299}$$

[10] Had we assumed a more general situation, where the particle is considered to come out of the cloud with a flat distribution in the μ' angle, we would have obtained

$$\langle\frac{\Delta E}{E}\rangle = \frac{4}{3}\beta^2$$

[11] Let us for now forget the presence of the weak magnetic field $\vec{B}_0$

Specific enthalpy is

$$w = \epsilon + \frac{P}{\rho} \tag{300}$$

Therefore

$$dw = d\epsilon + d\left(\frac{P}{\rho}\right) \tag{301}$$

On the other hand

$$\begin{aligned} dE &= dQ - PdV \\ &= \cancel{TdS} - PdV \\ \implies d\epsilon &= -Pd\left(\frac{1}{\rho}\right) \\ &= \frac{P}{\rho^2}d\rho \end{aligned} \tag{302}$$

$$\implies dw = \frac{P}{\rho^2}d\rho + \frac{dP}{\rho} - \frac{P}{\rho^2}d\rho = \frac{dP}{\rho} \tag{303}$$

Now we can use again the conservation of momentum and write:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{\nabla P}{\rho} \equiv \nabla\left[\epsilon + \frac{P}{\rho}\right] \tag{304}$$

Remember the energy density per unit volume

$$u = \frac{P}{\gamma_g - 1} = \rho\epsilon \implies \epsilon = \frac{1}{\gamma_g - 1}\frac{P}{\rho} \tag{305}$$

Therefore

$$w = \epsilon + \frac{P}{\rho} = \frac{\gamma_g}{\gamma_g - 1}\frac{P}{\rho} \tag{306}$$

So that conservation of energy equation can be cast in the following form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{\gamma_g}{\gamma_g - 1}\nabla\left(\frac{P}{\rho}\right) \tag{307}$$

Thermodynamics reminder

For an ideal gas the **internal energy** U is a function of temperature only:

$$\Delta U = nc_V \Delta T$$

where n is the number of moles, c_V is the specific heat at constant volume and T is the temperature. If take as reference values $T_0 = 0 \implies U_0 = 0$, then we can rename $U_1 = U$ and $T_1 = T$ and write $U = nc_V T$. Using the perfect gas equation of state $PV = nRT$ (R is the universal constant of gases) and the Mayer relations $c_P - c_V = R$ (with c_P the specific heat at constant pressure), along with the definition of the **adiabatic index**:

$$\gamma_g = \frac{c_P}{c_V}$$

we can write:

$$U = nc_V T = n\frac{R}{\gamma_g - 1}T = \frac{PV}{\gamma_g - 1} \implies \frac{U}{V} \equiv u = \frac{P}{\gamma_g - 1}$$

On the other hand **enthalpy** is defined as:

$$W = U + PV$$

If we divide by the volume V and use $u = \rho\epsilon$, where ρ is the density and ϵ is the specific energy (i.e. energy per unit mass), we obtain the specific enthalpy:

$$\frac{W}{V} = u + P = \rho\epsilon + P \implies \frac{W}{V\rho} = u + \frac{P}{\rho} \implies w = \epsilon + \frac{P}{\rho}$$

Last the first law of thermodynamics states that

$$dU = TdS - PdV$$

with S the entropy, which is conserved (so $dS = 0$). If we divide by mass we obtain the specific quantities, namely

$$d\epsilon = -Pd\left(\frac{1}{\rho}\right)$$

Therefore:

$$dw = d\epsilon + \frac{dP}{\rho} + Pd\left(\frac{1}{\rho}\right) = \frac{dP}{\rho}$$

Let us now rewrite the three conservation equations (mass, momentum and energy) making the assumptions of stationarity and 1-d

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0 \implies \rho u = \text{constant} \tag{308}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = \frac{1}{\rho}\frac{dP}{dz} \implies \frac{\partial}{\partial z}\left[\rho u^2 + P\right] = 0 \implies \rho u^2 + P = \text{constant} \tag{309}$$

$$u\frac{\partial u}{\partial z} + \frac{\gamma_g}{\gamma_g - 1}\frac{\partial}{\partial z}\left(\frac{P}{\rho}\right) = 0 \tag{310}$$

$$\frac{\partial}{\partial z}\left[\frac{1}{2}\rho u^3 + \frac{\gamma_g}{\gamma_g - 1}uP\right] = 0 \implies \frac{1}{2}\rho u^3 + \frac{\gamma_g}{\gamma_g - 1}uP = \text{constant} \tag{311}$$

Summarizing we have

$$\begin{cases} \rho u = \text{constant} \\ \rho u^2 + P = \text{constant} \\ \frac{1}{2}\rho u^3 + \frac{\gamma_g}{\gamma_g - 1} uP = \text{constant} \end{cases} \tag{312}$$

We have three unknowns and three equations. There is an immediate solution, which is the trivial one where all the quantities are constant.
There is also a non trivial solution which comes up if we assume

$$\frac{u}{c_s} > 1 \tag{313}$$

with $c_s = \gamma \frac{P}{\rho}$ the sound speed. The ratio $\frac{u}{c_s} = M$ is called the **Mach number**. If we fix the quantities in one region, ρ_1, u_1 and P_1, then solving the equations will lead us to

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{(\gamma_g + 1)M_1^2}{(\gamma_g - 1)M_1^2 + 2} \tag{314}$$

where

$$M_1 = \frac{u_1}{c_{s1}}, \quad \text{with } c_{s1}^2 = \gamma_g \frac{P_1}{\rho_1} \tag{315}$$

What about the pressure?

$$\frac{P_2}{P_1} = \frac{2\gamma_g M_1^2}{\gamma_g + 1} - \frac{\gamma_g - 1}{\gamma_g + 1} \tag{316}$$

And the temperature?

$$\frac{T_2}{T_1} = \frac{\left[2\gamma_g M_1^2 - (\gamma_g - 1)\right]\left[(\gamma_g - 1)M_1^2 + 2\right]}{(\gamma_g + 1)^2 M_1^2} \tag{317}$$

What happens if $M_1 \to +\infty$?

$$\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{\gamma_g + 1}{\gamma_g - 1} \underset{\gamma_g = \frac{5}{3}}{=} 4 \tag{318}$$

where we have assumed monoatomic gas. And

$$\frac{P_2}{P_1} = \frac{2\gamma_g}{\gamma_g + 1} M_1^2 \tag{319}$$

Note that the non trivial solution makes sense only if the Mach number M_1 is larger than one. This discontinuity is described by the parameter r, the compression factor, which depends on the adiabatic index γ_g and on the Mach number of the shock. Note also that we cannot have a compression factor larger than 4.

Solving the shock equations (1)

We have to solve the following system of equations:

$$\begin{cases} \rho_1 u_1 = \rho_2 u_2 \\ \rho_1 u_1^2 + P_1 = \rho_2 u_2^2 + P_2 \\ \frac{1}{2}\rho_1 u_1^3 + \frac{\gamma_g}{\gamma_g - 1} P_1 u_1 = \frac{1}{2}\rho_2 u_2^3 + \frac{\gamma_g}{\gamma_g - 1} P_2 u_2 \end{cases}$$

- **Derive** u_2/u_1:
  Let's start with the third one and divide by the first term $\frac{1}{2}\rho_1 u_1^3$:

$$1 + \frac{2}{\gamma_g - 1}\frac{\gamma_1 P_1}{\rho_1 u_1^2} = \frac{\rho_2}{\rho_1}\left(\frac{u_2}{u_1}\right)^3 + \frac{2\gamma_g}{\gamma_g - 1}\frac{u_2}{\rho_1 u_1^3} P_2$$

  Substitute $P_2 = \rho_1 u_1^2 - \rho_2 u_2^2 + P_1$ and use the first equation remembering that $\frac{\gamma_1 P_1}{\rho_1} = c_{S,1}^2$ and $\frac{u_1^2}{c_{S,1}^2} = M_1^2$

$$1 + \frac{2}{\gamma_g - 1}\frac{1}{M_1^2} = \left(\frac{u_2}{u_1}\right)^2 + \frac{2\gamma_g}{\gamma_g - 1}\left(\frac{u_2}{u_1}\right) - \frac{2\gamma_g}{\gamma_g - 1}\left(\frac{u_2}{u_1}\right)^2 + \frac{2}{\gamma_g - 1}\frac{1}{M_1^2}\left(\frac{u_2}{u_1}\right)$$

  Call $\frac{u_2}{u_1} = x$ and multiply every term by $(\gamma_g - 1)M_1^2$:

$$x^2 M_1^2(\gamma_g + 1) - 2x(\gamma_g M_1^2 + 1) + 2 + (\gamma_g - 1)M_1^2 = 0$$

  Which gives us the trivial solution $\frac{u_2}{u_1} = 1$ and the non trivial one given by

$$\frac{u_2}{u_1} = \frac{(\gamma_g - 1)M_1^2 + 2}{(\gamma_g + 1)M_1^2}$$

Solving the shock equations (2)

- **Derive** P_2/P_1:
  We start again with the third equation and divide by the second term $\frac{2\gamma_g}{\gamma_g - 1} u_1 P_1$:

$$\frac{\rho_1 u_1^3(\gamma_g - 1)}{2\gamma_g u_1 P_1} + 1 = \frac{\rho_2 u_2^3(\gamma_g - 1)}{2\gamma_g u_1 P_1} + \frac{u_2}{u_1}\frac{P_2}{P_1}$$

  Therefore:

$$\frac{P_2}{P_1} = \frac{u_1}{u_2}\left(\frac{2 + M_1^2(\gamma_g - 1)}{2}\right) - \frac{M_1^2(\gamma_g - 1)}{2}\frac{u_2}{u_1}$$

  Substituting the ratio u_2/u_1 found in the first point we obtain:

$$\frac{P_2}{P_1} = \frac{2\gamma_g M_1^2}{\gamma_g + 1} - \frac{\gamma_g - 1}{\gamma_g + 1}$$

Solving the shock equations (3)

- **Derive** T_2/T_1:
  using the perfect gas equation of state we have that

$$P = nk_BT\frac{m_p}{m_p} \implies \frac{P}{T\rho} = constant$$

Therefore:

$$\frac{P_1}{T_1\rho_1} = \frac{P_2}{T_2\rho_2} \implies \frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{P_2}{P_1}\frac{u_2}{u_1}$$

Substituting the first two relations we finally find:

$$\frac{T_2}{T_1} = \frac{\left[2\gamma_g M_1^2 - (\gamma_g - 1)\right]\left[(\gamma_g - 1)M_1^2 + 2\right]}{(\gamma_g + 1)^2 M_1^2}$$

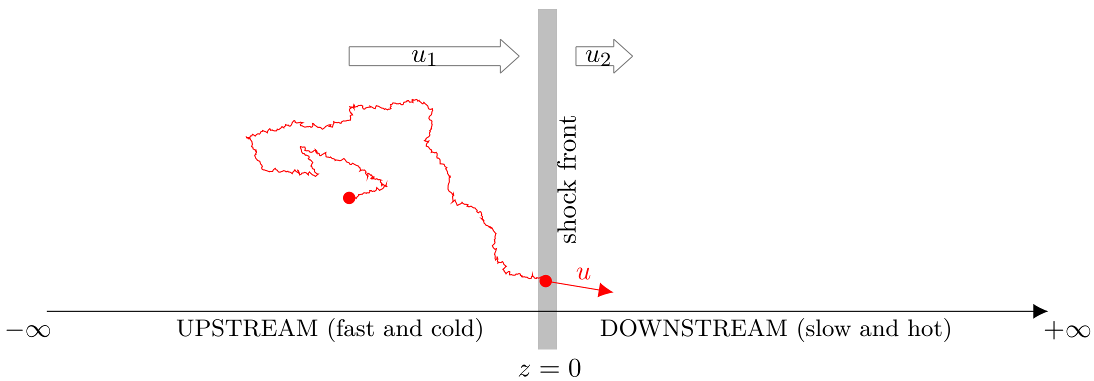
We have found that if $M > 1$ the plasma is moving with a velocity larger than the sound speed, which is the velocity of the perturbations. We'll have a discontinuity and if we are sitting on that discontonuity, then we can see an upcoming wave with a velocity u_1, but on the other side this velocity will be changed to $u_2 \sim \frac{1}{4}u_1$. We have obtained that the plasma has been slowed down and made denser. But energy has to be conserved and it can only go into **heat**. In fact, if we look at the ratio $\frac{T_2}{T_1}$ we find that

$$\frac{T_2}{T_1} = \frac{2\gamma_g(\gamma_g - 1)}{(\gamma_g + 1)^2}M_1^2 \tag{320}$$

Therefore

$$\begin{aligned} k_BT_2 &= k_BT_1\frac{2\gamma_g(\gamma_g - 1)}{(\gamma_g + 1)^2}M_1^2 \\ &= \cancel{k_BT_1}\frac{2(\gamma_g - 1)}{(\gamma_g + 1)^2}u_1^2\frac{\rho_1}{\cancel{k_BT_1}n_1} \\ &= \frac{2(\gamma_g - 1)}{(\gamma_g + 1)^2}m_pu_1^2 \\ &= \frac{3}{16}m_pu_1^2 \end{aligned} \tag{321}$$

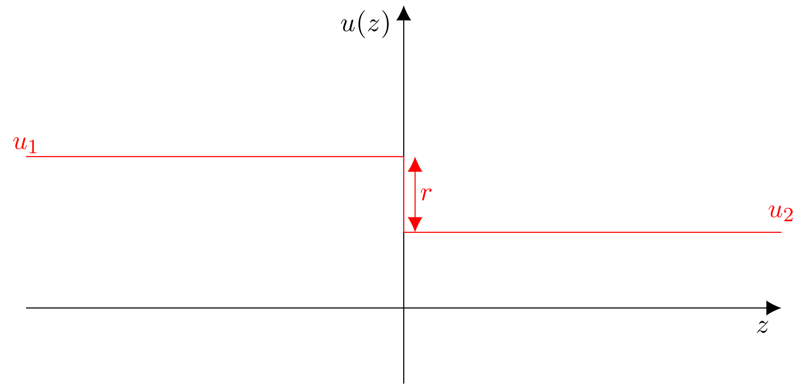
where we have used $\rho_1 = n_1m_p$, $M_1^2 = u_1^2/c_{s,1}^2$, $c_{s,1}^2 = \gamma P_1/\rho_1$, and $P_1 = n_1k_BT_1$.
We have obtained that the energy of the particle downstream is a large fraction of the kinetic energy of the particle upstream. Kinetic energy is converted into thermal energy! We call this surface a **strong shock wave**. It behaves as a heater transforming kinetic into thermal energy.



In the downstream region density and temperature increase: the kinetic energy of the upstream particles is turned into thermal energy of the downstream particles. Shock waves slow down the plasma and heat it up. This is what happens for example in a supernova explosion ($M \sim 1000$) or in GRB. The plasma crosses shocks and heats up until it starts radiating in X-ray.

As a curiosity, shocks produced in the astrophysical context are different from those produced by military planes. The last ones are collisional shock waves, whereas the astrophysical ones are collisionless.

Shocks are important for particle acceleration. When a plasma moves with a velocity greater than the sound speed, it develops a shock, which is a region of discontinuity. Usually we have relatively strong shocks, with a Mach number of order of 1000.



When we are sitting on the discontinuity region, the upstream region is characterized by fast and cold plasma, whereas the downstream is characterized by slow and hot plasma. The velocity of the shock wave is of order

$$v_{\text{shock}} \sim 10000\text{km/s} \tag{322}$$

If we are sitting on the shock wave, we see the interstellar medium coming towards us with this same velocity from both the upstream and the downstream.
If we are in the downstream, we see it move toward us with speed $u_1 - u_2$. In the upstream we see again the interstellar medium moving toward us with opposite velocity. In both cases we have head-on collisions.
Put a test particle (not affecting the system) in the upstream, where it has energy E. The particle diffuses and crosses the shock. To calculate its energy once it reaches the downstream, E_d, we can use Lorentz transformation

$$E_d = \gamma E(1 + \beta\mu) \tag{323}$$

where $0 \leq \mu \leq 1$ and $\beta = \frac{u_2 - u_1}{c}$.
The same will happen to a particle going from downstream to upstream. What changes is the sign of the angle μ':

$$E_u = \gamma E_d - \beta E_d \gamma \mu' = \gamma^2 E(1 + \beta\mu)(1 - \beta\mu') \tag{324}$$

with $-1 \leq \mu' \leq 0$.
What we have obtained is that the final energy, after a cycle (upstream -> downstream -> upstream), $\Delta E/E > 0$. There are no configurations leading to energy decrease: ΔE is bound to be positive.
Let's calculate the flux of particles going in the direction from upstream to downstream:

$$J = \int_0^{+1} \frac{N}{4\pi} d\Omega v\mu = \frac{Nv}{4} \tag{325}$$

The probability of having a scattering with pitch angle μ is

$$P(\mu) = \frac{ANv\mu}{\frac{Nv}{4}} = 4A\mu \tag{326}$$

Requiring

$$\int_0^1 d\mu 4A\mu = 1 \implies A = \frac{1}{2} \implies P(\mu) = 2\mu \tag{327}$$

It's the same in the other direction and we can write

$$\begin{aligned}\langle \frac{E_u - E}{E} \rangle_{\mu,\mu'} &= -\int_0^1 d\mu \int_{-1}^0 d\mu' P(\mu)P(\mu') \left[\gamma^2(1+\beta\mu)(1-\beta\mu') - 1\right] \\ &= \frac{4}{3}\frac{u_2 - u_1}{c} \\ &= \frac{4}{3}\beta\end{aligned} \tag{328}$$

By eliminating all configurations in which E decreases, we have transformed a process from being second order in β to first order. If $u_1 \sim 1000$km/s and $u_2 = \frac{1}{4}u_1$, we have that $\beta = \frac{u_1 - u_2}{c} \sim 10^{-2} - 10^{-3}$, which is much higher than v_A. The gain is enormous! Of course we need many cycles to accelerate particles to energies we observe in cosmic rays. Remember that these processes have been understood between the end of 1970s and the beginning of 1980s (see Bell's paper of 1978). But cosmic rays have an energy spectrum, which is a power law. How does it come?

Note that as long as the particle remains downstream or upstream it just moves around, with no gains in energy.

ii.1 Probability of return and particle spectrum at accelerator

Let's put ourselves in the reference system of the immediate downstream of the shock. What is the flux of particles entering the upstream?

$$\phi_{\text{out}} = \int (u_2 + v\mu) f_0 d\mu \tag{329}$$

where $u_2 + v\mu < 0 \overset{v \sim 1}{\implies} \mu < -u_2 \implies -1 < \mu < -u_2$. So

$$\phi_{\text{out}} = -\int_{-1}^{-u_2} d\mu f_0(u_2 + \mu) = \frac{1}{2} f_0(1-u_2)^2 \tag{330}$$

and in a similar way

$$\phi_{\text{in}} = \int_{-u_2}^{1} d\mu f_0(u_2 + \mu) = \frac{1}{2} f_0(1+u_2)^2 \tag{331}$$

What is the probability that the particle returns to the shock? Let's call it the probability of return, P_{ret}

$$P_{\text{ret}} = \frac{\phi_{\text{out}}}{\phi_{\text{in}}} = \frac{(1-u_2)^2}{(1+u_2)^2} = 1 - 4u_2 \quad \text{if} \quad u_2 \ll 1 \tag{332}$$

with $u_2/c \sim 10^{-2}$. So there is a chance particles will return to the shock. This is also the reason why we do not expect the particles to be accelerated in the same way. Therefore we have a spectrum of energy, a distribution. Here we are developing a statistical approach. If a particle starts with energy E_0, after one cycle

$$\frac{E_1 - E_0}{E_0} = \frac{4}{3}\beta \implies E_1 = E_0\left(1 + \frac{4}{3}\beta\right) \tag{333}$$

After two crossing

$$E_2 = E_0\left(1 + \frac{4}{3}\beta\right)^2 \tag{334}$$

and after k cycles

$$E_k = E_0\left(1 + \frac{4}{3}\beta\right)^k \tag{335}$$

Suppose we have N_0 particles with energy E_0, we will have

$$N_1 = N_0 P_{\text{ret}} \quad \dots \quad N_k = N_0 P_{\text{ret}}^k \tag{336}$$

Let's calculate

$$\frac{E_k}{E_0} = \left(1 + \frac{4}{3}\beta\right)^k \implies \log\left(\frac{E_k}{E_0}\right) = k \log\left(1 + \frac{4}{3}\beta\right) \tag{337}$$

and

$$\frac{N_k}{N_0} = P_{\text{ret}}^k \implies \log\left(\frac{N_k}{N_0}\right) = k \log\left(P_{\text{ret}}\right) \tag{338}$$

This means that

$$\frac{\log\left(\frac{E_k}{E_0}\right)}{\log\left(1 + \frac{4}{3}\beta\right)} = \frac{\log\left(\frac{N_k}{N_0}\right)}{\log\left(P_{\text{ret}}\right)} \tag{339}$$

In the limit $\beta \ll 1$ and $u_2 \ll 1$, we can Taylor expand[12]

$$\frac{N(E_k)}{N_0} = \left(\frac{E_k}{E_0}\right)^{-\gamma} \tag{340}$$

with $\gamma = \frac{3}{r-1}$ with r the ratio $r = u_1/u_2$, called the compression factor. We indeed obtain that the number of particles that reach at least energy E_k is a power law

$$N(> E_k) = N_0 \left(\frac{E_k}{E_0}\right)^{-\gamma} \tag{341}$$

Note that the index γ depends only on r.
The results seems independent of details, of the microphysics. Note that if D_{zz} is small we can cross the shock many times and reach high energies. Diffusion plays an important part in how much particles get accelerated.
It is useful to introduce the differential spectrum $n(E)dE$, the number of particles in the bin dE:

$$n(E)E \sim E^{-\gamma} \implies n(E) \sim E^{-\gamma-1} \tag{342}$$

with $\gamma + 1 = \frac{3}{r-1} + 1 = \frac{3+r-1}{r-1} = \frac{r+2}{r-1}$. For a strong shock $\gamma + 1 = 2$.

Where's the microphysics?

We have found here an extraordinary result: the differential energy spectrum of our particles is a simple power law, $n(E) \sim E^{-\gamma-1}$, equivalently $n(E) \sim E^{-2}$ for relativistic particles. **The spectrum only depends on the compression factor!** The result is completely independent of the diffusion coefficient, that is to say of the microphysics. This is somehow reassuring, because it is difficult to keep track of the details of a process taking place at distances kpc away from us.
On the other hand, the efficiency of the process, the possibility to accelerate particles to a high enough energy, depends dramatically on the diffusion coefficient.
Finally the presence of shock waves in astrophysics is pretty ubiquitous. We find them in **supernovae**, **pulsar wind nebulae, gamma ray bursts**. Pay attention that we have only described non relativistic shocks, but the core physics remains the same.

[12]Remember the Taylor expansion for the logarithm

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Here we just stop at the first order:

$$\frac{\log(1-4u_2)}{\log(1+\frac{4}{3}(u_1-u_2))} = \frac{-4u_2}{\frac{4}{3}(u_1-u_2)} = -\frac{3\frac{u_2}{u_2}}{\frac{u_1}{u_2} - \frac{u_2}{u_2}} = -\frac{3}{r-1} = -\gamma$$

Let's put ourselves as usual on the reference frame of the shock and introduce the particle distribution function

$$f = f(z, t, p) \tag{343}$$

The number of the particles in a bin will be

$$n(z, t, p)dp = f(z, t, p)4\pi p^2 dp \tag{344}$$

The equation the distribution f satisfies is

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial z} - \frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial f}{\partial p} = \frac{\partial}{\partial z}\left[D_{zz}\frac{\partial f}{\partial z}\right] + Q \tag{345}$$

where we have introduced the **injection term** Q. If there are no injection terms or boundary conditions, the only solution we can find is the null one, since this is a linear partial differential equation in time, space and momentum. Therefore, since it is linear, we cannot find something if we do not inject it.

Pay attention that in our system the shock is a collisionless one and it has a thickness (it cannot be a mathematical surface). Its thickness is comparable to the Larmor radius of the thermal particles in the system, the ones that are forming the shock. For typical shock velocities of few thousands kilometers per second, we obtain $r_L \sim 10^8$ cm. This is not large for astrophysical contexts: they can truly be thought of as surfaces.
Let's think for a second at the gas behind the shock, i.e. the downstream. Here we will find a distribution of momenta, a Maxwellian to be precise, since the gas is thermalized. At the tail of the momentum distribution (where momentum are a little bit larger than average), and if we are very close to the shock surface, these particles have larger Larmor radii and these are the particles that have a finite chance to find themselves at the other side of the shock. At this point, they start gaining energy: they no longer belong to the Maxwellian, *they have been injected*. These particles are the seeds of the acceleration. If we repeat the same chain of thoughts for particles that are farther away from the shock surface, it becomes less probable that they will ever cross the shock, because it would require very large Larmor radii. This is the reason why we can assume that **particles are injected at the shock**.

$$Q(z, t, p) = \delta(z)\delta(p - p_{inj})\underbrace{\frac{\eta N_{inj} u_1}{4\pi p_{inj}^2}}_{Q_0} \tag{346}$$

where Q_0 is a fraction of the flux of particles crossing the shock per unit volume in phase space.

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial z} - \frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial f}{\partial p} = \frac{\partial}{\partial z}\left[D_{zz}\frac{\partial f}{\partial z}\right] + \delta(z)\delta(p - p_{inj})Q_0 \tag{347}$$

Let's see what happens if we focus on the upstream only and assume that the system is **stationary** ($\frac{\partial f}{\partial t} = 0$). Also, we expect $\frac{du}{dz} = 0$, except for the location of the shock. So in the upstream:

$$u\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = 0 \implies \frac{\partial}{\partial z}\left[uf - D\frac{\partial f}{\partial z}\right] = 0 \tag{348}$$

Therefore $uf - D\frac{\partial f}{\partial z} = k$ is the total flux[13] and it is constant. Since the flux should be zero at infinity we have $k = 0$ (it would be unphysical to have accelerated particles at the upstream infinity)

$$uf - D\frac{\partial f}{\partial z} = 0 \tag{349}$$

[13] $D\frac{\partial f}{\partial z}$ has the physical meaning of a current, it represents the **diffusive flux**. On the other hand we have the **advective flux** which is uf. Therefore $uf - D\frac{\partial f}{\partial z}$ is the total flux.

Suppose the following form for the distribution function

$$f = f_0 \exp\left[\alpha z\right] \tag{350}$$

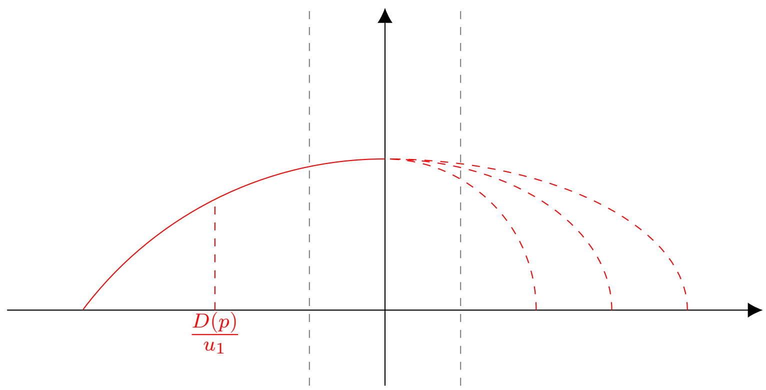
therefore applying (349) we obtain

$$u f_0 \exp\left[\dots\right] - D f_0 \alpha \exp\left[\dots\right] = 0 \implies \alpha = \frac{u_1}{D} \tag{351}$$

So in the upstream the distribution function is

$$f(z,p) = f_0 \exp\left[\frac{u_1 z}{D(p)}\right] \tag{352}$$

where f_0 is the distribution function at the shock.



$D(p)/u_1$ is called the typical **diffusion length** of the plasma. The particles are trying to get away from the shock by diffusion and the plasma is pushing them back towards the shock. Therefore, there is a typical distance at which these two processes counterbalance.
In the downstream the only solution that makes sense, given the assumption on stationarity, is that the distribution of the particles is constant. The particle density in the downstream cannot diverge at the downstream infinity (it would be unphysical), so it has to decrease or stay constant. If it decreases, since diffusion and advection are working to take the particles away, it would no longer be stationary.

$$f = f_0 \tag{353}$$

Note that f is continuous across the shock: all the discontinuities are properties of the plasma. In fact, the particles we are considering are particles whose Larmor radius, by construction, is larger than the shock. Therefore, these particles cannot know about the discontinuity. Dynamical and thermodynamical discontinuities are properties of the plasma.
Now let's integrate the transport equation between 0^- and 0^+ to find f_0:

$$0 - \frac{1}{3}(u_2 - u_1) p \frac{\partial f_0}{\partial p} = \underbrace{D\frac{\partial f}{\partial z}|_2}_{0} - \underbrace{D\frac{\partial f}{\partial z}|_1}_{u_1 f_0} + Q_0 \delta(p - p_{inj}) \tag{354}$$

where the integral of $\frac{\partial f}{\partial z}$ is zero since f is continuous across the shock. This implies:

$$u_1 f_0 - \frac{1}{3}(u_2 - u_1) p \frac{\partial f_0}{\partial p} = Q_0 \delta(p - p_{inj}) \tag{355}$$

The solutions make sense if $p > p_{inj}$, since in all configurations we can only increase the energy and we cannot have momenta lower than the injected ones (moreover, we can use the $\delta-$ function as a boundary condition), therefore

$$u_1 f_0 = \frac{1}{3}(u_2 - u_1) p \frac{\partial f_0}{\partial p} \tag{356}$$

We can rewrite this as

$$\begin{aligned} \frac{df_0}{f_0} &= -\frac{3u_1}{u_1 - u_2}\frac{dp}{p} \\ \log f_0 &= -\frac{3u_1}{u_1 - u_2}\log p \\ f_0(p) &= p^{-\frac{3u_1}{u_1-u_2}} = p^{-\frac{3u_1}{u_1-u_2}} \\ f_0(p) &= \frac{3u_1}{u_1 - u_2}\frac{N_{inj}\eta}{4\pi p_{inj}^2}\left(\frac{p}{p_{inj}}\right)^{-\frac{3r}{r-1}} \\ f(p) &\propto p^{-\frac{3r}{r-1}} \end{aligned} \tag{357}$$

We have derived again that $En(E) \sim E^{-\gamma}$, but $n(E)dE = 4\pi p^2 f(p) dp$, so for relativistic particles ($E = p$) and we can write

$$n(E) \sim p^2 f(p) \sim p^2 p^{-\frac{3r}{r-1}} \sim E^{\frac{2r-2-3r}{r-1}} \sim E^{-\frac{r+2}{r-1}} \tag{358}$$

If $r = 4$ we have $n(E) \sim E^{-2}$.
If the particles are not relativistic, then $E \neq p$, but $E = \frac{p^2}{2m}$, and we find $n(E) \sim E^{-3/2}$.

Energy spectrum of non relativistic particles

The number of particles with energy E is given by

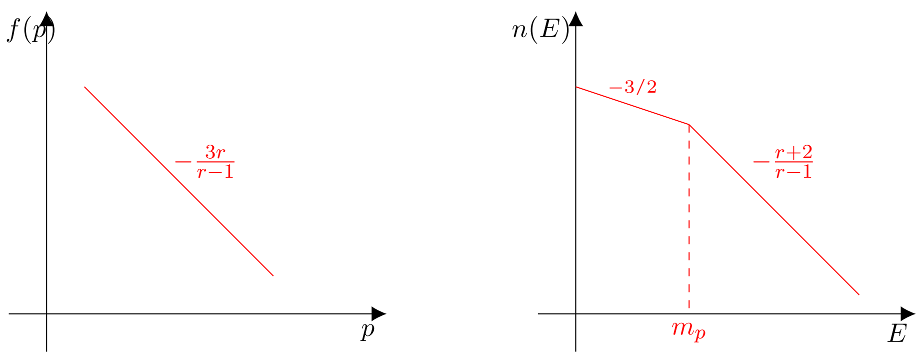
$$n(E)dE = 4\pi p^2 f_0(p)\left(\frac{dp}{dE}\right)dE$$

We have that $f_0(p) \sim p^{-\frac{3r}{r-1}}$ and the momentum-energy relation for non relativistic particles is $p \sim \sqrt{E}$. Therefore:

$$n(E) \propto p^2 p^{-\frac{3r}{r-1}}\frac{dp}{dE} = E E^{-\frac{1}{2}\frac{3r}{r-1}} E^{-\frac{1}{2}} = E^{-\frac{2r+1}{2(r-1)}} \xrightarrow[r=4]{} E^{-3/2}$$

So the spectrum of particles accelerated in diffusive shock acceleration is a power law in terms of momentum, and a broken power law in terms of energy, with the break at the particle mass.

$$f(p) \sim p^{-\frac{3r}{r-1}} \tag{359}$$



D disappears from the game. The slope only depends on the compression factor r. The number of particles being accelerated depends on the injection factor. However, we do not know how many particles are being accelerated: this depends on the microphysics.
We have seen that the spectrum in energy in case of strong shock goes like $n(E) \sim E^{-2}$. Moreover, we have assumed that the system is stationary, so what is the maximum energy that we expect? Assuming the system stationary it should be infinite (we cannot have configurations in which energy decreases), which is absurd. Let's calculate the total energy of the system, where we focus on the relativistic particles only:

$$\epsilon_{tot} = \int_{E_{min}=m_p c^2}^{E_{max}} dEn(E)E = \int_{E_{min}}^{E_{max}} dEE^{-2}E = \log\left(\frac{E_{max}}{E_{min}}\right) \tag{360}$$

Where do cosmic rays are taking energy from? Energy must be conserved! We are converting the kinetic energy of the plasma. Locally the energy density is ρu^2, and we can always choose E_{max} close to ρu^2 (however E_{max} depends on the microphysics of the transport). The problem here is that this is a **text particle theory**, where cosmic rays are assumed to behave as test particles and as such they cannot possibly affect the system. This has as a consequence that the accelerated particles can reach energies comparable to those of the plasma itself.

1. This is a signal that, in practice, we have to develop a non-linear theory, **we cannot have test particles**.

2. If we assume E_{max} finite the system cannot be stationary: the energy can only increase. The only way to avoid this conclusion is to allow the particles to leave the system:

$$f(z_0, p) = 0 \tag{361}$$

   This condition is known as **free escape boundary condition**. In this case we would obtain that there is an exponential decay for the maximum value of momentum.

> **The problem of maximum energy**
>
> Why should we have a maximum energy in the first place?
> The answer to this is that our system is not truly stationary: if our system is a thousand years old, for example, it cannot have been accelerating for longer than that. Therefore we need to compare the age of our system with the **acceleration time**, which what we are going to do in the following section.

III. Dynamical reaction of accelerated particles

To make an estimate of the acceleration time we need to know how much energy we gain per each crossing and how much time each crossing takes. The number of particles that flows in a certain direction, say the upstream, can be written as

$$N_\Sigma = \frac{n_{CR} v}{4} \Sigma \tau_1 \tag{362}$$

where Σ is a generic surface that we are considering.
But, on the other side

$$N_\Sigma = n_{CR} \Sigma \frac{D_1}{u_1} \tag{363}$$

where $\frac{D_1}{u_1}$ is the time needed to fill one diffusion length in upstream.
Therefore we can write

$$\tau_1 = \frac{4D_1}{vu_1}, \quad \text{and} \quad \tau_2 = \frac{4D_2}{vu_2} \tag{364}$$

So the time necessary to fill a diffusion length volume upstream abd downstream is the sum of two times.
The fraction of energy I gain is equal to $\frac{\Delta E}{E} = \frac{4}{3}\frac{u_1 - u_2}{c}$, so

$$\tau_{acc} = \frac{E}{\frac{\Delta E}{\Delta t}} = \frac{E}{\Delta E}\tau = \frac{3}{4}\frac{c}{u_1 - u_2}\frac{4}{v}\left[\frac{D_1}{u_1} + \frac{D_2}{u_2}\right] \overset{v \sim c}{=} \frac{3}{u_1 - u_2}\left[\frac{D_1}{u_1} + \frac{D_2}{u_2}\right] \tag{365}$$

Therefore

$$\tau_{acc} = \frac{3}{u_1 - u_2}\left[\frac{D_1}{u_1} + \frac{D_2}{u_2}\right] \tag{366}$$

Without approximations we would have obtained

$$\tau_{acc} = \frac{3}{u_1 - u_2}\int \frac{dp}{p}\left[\frac{D_1}{u_1} + \frac{D_2}{u_2}\right] \tag{367}$$

Now we can compare the acceleration time with the age of our system. Remember that in a galaxy one can estimate the diffusion coefficient by measuring secondary nuclei (borum, berillium and lithium). Approximately

$$D(E) = 3 \times 10^{28}\text{cm}^2/\text{s } \text{E}_{\text{GeV}}^{1/2} \tag{368}$$

Let's calculate τ_{acc} for a supernova where $u = 10000\text{km/s}$

$$\tau_{acc} \sim \frac{D}{v_{shock}^2} = \frac{3 \times 10^{28}\text{cm}^2/\text{s } \text{E}_{\text{GeV}}^{1/2}}{10^{18}\text{cm}^2/\text{s}^2} = 3 \times 10^{10}\text{E}_{\text{GeV}}^{1/2} \sim 1000\text{yr}\text{E}_{\text{GeV}}^{1/2} \tag{369}$$

If we consider Tycho's supernova (1572), we see particles being accelerated up 100TeV. How is this possible? What are we missing? The only way is to reduce the diffusion coefficient D! Otherwise the acceleration time is orders of magnitude too long. The main physical challenge is how to reduce D. Cosmic rays can perturb the system and decrease D. The problem is non-linear anymore.
We have studied conservation laws

$$\frac{\partial}{\partial z}\left[\rho u^2 + P_{gas}\right] = 0 \implies \rho u^2 + P_{gas} = \text{constant} \tag{370}$$

What if our particles affect the system? We have $\rho \sim$eV cm^{-3}, $E_{CR} = 1\text{GeV}$, $n = 1\text{cm}^{-3}$ and $n_{CR} = 10^{-9}\text{cm}^{-3}$ (cosmic rays can be neglected from the point of view of conservation of mass). In the presence of cosmic rays the momentum conservation equation becomes

$$\rho u^2 + P_{gas} + P_{CR} = \text{constant} \tag{371}$$

What happens in the upstream? If we are sitting at the upstream infinity (we call it with a subscript 0) there are no cosmic rays:

$$\begin{aligned} \rho_0 u_0^2 + P_{gas,0} &= \rho_1 u_1^2 + P_{gas,1} + P_{CR} \\ 1 + \cancel{\frac{P_{gas,0}}{\rho_0 u_0^2}} &= \frac{\rho_1 u_1^2}{\rho_0 u_0^2} + \cancel{\frac{P_{gas,1}}{\rho_0 u_0^2}} + \frac{P_{CR}}{\rho_0 u_0^2} \\ 1 &= \frac{u_1}{u_0} + \xi_{CR} \end{aligned} \tag{372}$$

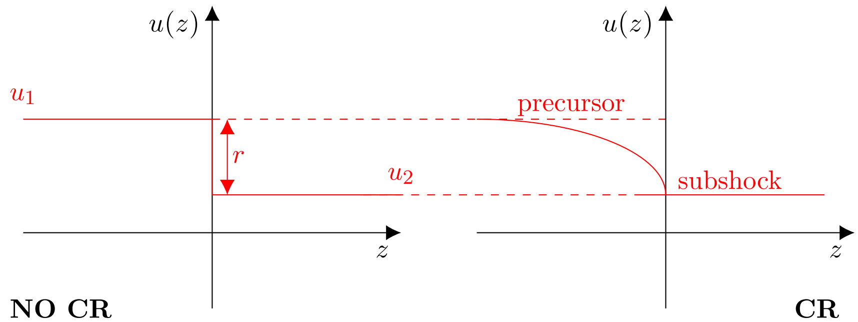
In the 0 the cosmic rays are suppressed and we have used

$$\frac{P_{gas,0}}{\rho_0 u_0^2} = \frac{c_S^2}{\gamma_g u_0^2} = \frac{1}{\gamma_g M_0^2} \overset{M_0 \gg 1}{\longrightarrow} \ll 1 \tag{373}$$

and $\rho_1 u_1 = \rho_0 u_0$.
If $\xi_{CR} = 0$ we are not accelerating particle. If $\xi_{CR} \neq 0$

$$\frac{u_1}{u_0} = 1 - \xi_{CR} \tag{374}$$

So when we are accelerating particles effectively, plasma in the upstream slows down. There are no such things as test particles.



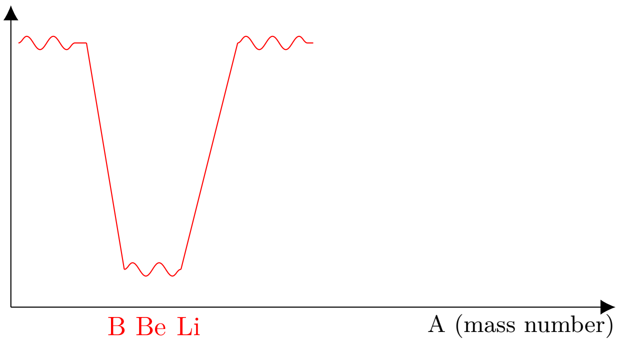
We have two processes:

1. **Dynamical action of accelerated particles**: imposing mass and momentum conservation we have found that
$$\xi_{CR} \equiv \frac{P_{CR,1}}{\rho_0 u_0^2} = 1 - \frac{u_1}{u_0} \tag{375}$$
When particles are accelerated, the plasma has to slow down. At low energies particles will have a smaller diffusion coefficient than a high energy particle. It feels a compression factor r_1, whereas a high energy particle feels the compression factor r_2. So **the compression factor r depends on momentum**. **The power spectrum cannot be a perfect power law**. At maximum $r = 4$, for lower r, the spectrum becomes steeper.

2. **Magnetic implication**
$$D(p) = \frac{1}{3} r_L(p) v \frac{1}{\mathcal{F}(k)} \implies \mathcal{F}(k) \sim \frac{\delta B^2}{B_0^2} \left(k_{res} = \frac{1}{r_L} \right) \tag{376}$$
The only way to decrease acceleration time is to decrease D. So we have to increase δB, we have to create large perturbations. Remember that in non linear theories the parameters u and D of the transport equation become dependent on f itself. The action of CR is to slow down the plasma.

IV. Basics of Supernova paradigm for CR

I. How much energy do we need to accelerate CR?



We start with the following question: why are the quantities of B, Be and Li so low in the Universe and yet we observe them in cosmic rays? They are not produced in the Big Bang nucleosynthesis. The Universe is expanding too fast to produce them. The curve then goes up, because they are synthesized from hydrogen and helium by stars. They also produce B, Be and Li, but they as well easily destroyed.
Still B, Be and Li are observed in the cosmic rays. They can be produced through **spallation** processes like

$$A + N \to (A-1) + N \tag{377}$$

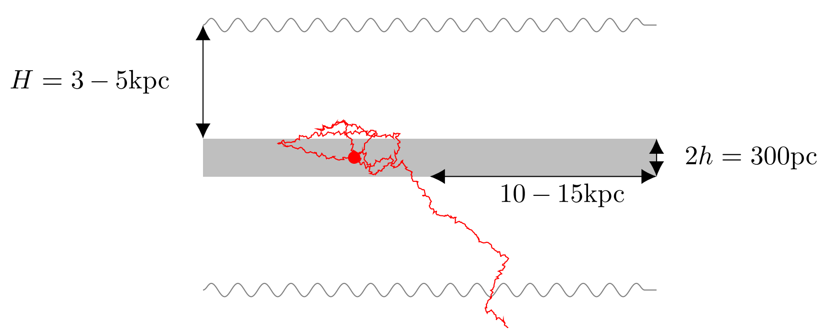
where A is the atomic number of the target nucleus.
For example, a carbon nucleus can be transformed into lighter elements. The cross section[14] of spallation is

$$\sigma_{sp} \simeq 45\text{mb}A^{0.7} \tag{378}$$

In cosmic rays we can measure a ratio between B and C of $\frac{B}{C} \sim 0.3$ at the energy of 10GeV/nucleon (remember that it drops to 10^{-7} in the interstellar medium).
The spallation timescale is roughly given by

$$\tau_{sp} = \frac{1}{\bar{n}\sigma_{sp}c} \tag{379}$$

where $\bar{n}$ is the mean density in the galaxy



Thee is also a region in our toy model of a galaxy surrounding the galaxy and called halo, where the density is negligible, but which is highly magnetized

$$n_{disk} \equiv n_d = 1\text{cm}^{-3} \text{ and } n_{halo} \equiv n_H \ll n_d \tag{380}$$

[14]Remember that

$$1\text{barn} = 10^{-24}\text{cm}^2$$

Since cosmic rays probe both, halo and disk, $\bar{n}$ should be an average of the two. Calling h the half thickness of the disk, we can write

$$\bar{n} = \frac{n_d \pi R_d^2 h + n_H \pi R_d^2 H}{\pi R_d^2 H} \overset{n_H \ll n_d}{\sim} n_d \frac{h}{H} \ll n_d \tag{381}$$

Therefore if $\tau_{sp} \sim \tau_{esc}$, where τ_{esc} is the time it takes for cosmic rays to leave the galaxy, then we would expect an equal abundance of B and Be. Since the ratio is 1/3, we can assume

$$\tau_{sp} \sim 3\tau_{esc} \tag{382}$$

So the escape time is

$$\tau_{esc} \simeq \frac{1}{3} \frac{1}{\bar{n}\sigma_{sp} c} = 90\text{Myr} \left(\frac{H}{3\text{kpc}} \right) \tag{383}$$

at an energy of 10GeV per nucleon. Notice that if the motion were ballistic, we would have obtained $\tau_{esc} \sim \frac{H}{c}$. Given that the total energy density of cosmic rays per unit volume is similar to that of the interstellar medium or the CMB or the energy density of the magnetic field:

$$\omega_{CR} \sim 1\text{eVcm}^{-3} \tag{384}$$

What can support this energy? Take the cosmic rays in a galaxy

$$E^{gal}_{TOT,CR} = \omega_{CR} \pi R_d^2 2H \tag{385}$$

Given a source of energy E_{source}, which occurs at a rate $\mathcal{R}$. To provide the energy necessary for cosmic rays it should happen that

$$\xi_{CR} E_{source} \mathcal{R} \tau_{esc} = \omega_{CR} \pi R_d^2 2H \tag{386}$$

where we have use ξ_{CR} because only a fraction of the source energy is passed to cosmic rays. So

$$\xi_{CR} = \frac{\omega_{CR} \pi R_d^2 2H}{E_{source} \mathcal{R} \tau_{esc}} \tag{387}$$

If ξ is less than one, then we have a possible source.

II. SNe as a source for CRs

Can a supernova be a source? We have that $E_{source} = 10^{51}$erg (only the kinetic part, not the gravitational which mostly goes to neutrinos), $\mathcal{R}_{SN} = \frac{1}{30\text{yr}}$ and we obtain $\xi_{CR} = 3\%$. It could be a good choice.

A supernova can be of two types:

- **Type IA**: these are derived from accretion induced explosion of a white dwarf (after it reaches the Chandrasekhar limit). An example is Cassiopea A (16 august 1680), roughly 11000ly from us. Supernovae IA are like nuclear bomb explosion, they leave no remnants.
- **Type IIA**: they leave neutron stars as remnants.

The energy of a supernova ejecta is of order $E_{SN} \sim 10^{51}$erg[15]. If $E_{SN} = \frac{1}{2} M_{ej} v_{ej}^2$ (this is the case because we are in the non relativistic regime), we can write

$$v_{ej} \sim 10^9 \text{cm s}^{-1} \left(\frac{E_{SN}}{10^{51}} \text{erg} \right)^{1/2} \left(\frac{M_{ej}}{1M_\odot} \right)^{-1/2} \tag{388}$$

[15]Remember that 1erg $= 10^{-7}$J

For a comparison

$$c_S = \left(\gamma_g \frac{P}{\rho}\right)^{1/2} = 10^6 \text{cm s}^{-1} \left(\frac{T}{10^4 \text{K}}\right)^{1/2} \tag{389}$$

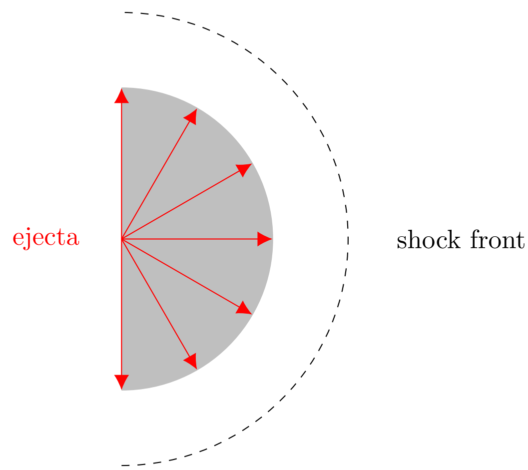
and[16]

$$v_A = \frac{B_0}{\sqrt{4\pi\rho}} = \frac{B_0}{\sqrt{4\pi n m_p}} \sim 2 \times 10^5 \text{cm s}^{-1} \left(\frac{n}{1\text{cm}^{-3}}\right)^{-1} \left(\frac{B_0}{\mu\text{G}}\right) \tag{390}$$

We have

- **Sonic Mach number**:
$$M_S = \frac{v_{ej}}{c_S} = 10^3 E_{51}^{1/2} M_{ej,\odot}^{-1/2} T_4^{-1/2} \tag{391}$$
- **Alfven Mach number**:
$$M_A = \frac{v_{ej}}{v_A} = 5 \times 10^3 E_{51}^{1/2} M_{ej,\odot}^{-1/2} n_1^{1/2} B_{\mu\text{G}}^{-1} \tag{392}$$

These are high numbers: we have a shock front!



ii.1 Ejecta-dominated and Sedov-Taylor phases

The shock front separates the unshocked interstellar medium from the shocked interstellar medium. Remember that locally the shock looks like 1d. Let's think in terms of masses: the shock is moving through the interstellar medium and accumulates mass:

$$M_{acc} = \rho_{ISM} v_{ej} 4\pi R_{shock}^2 t \tag{393}$$

As long as the Mach number is not too high we are not changing much the dynamics. Therefore:

$$\begin{aligned} M_{acc} &= \rho_{ISM} v_{ej} 4\pi (v_{ej} t)^2 t \\ &= \rho_{ISM} v_{ej}^3 t^3 4\pi \\ &= 6.2 \times 10^{26} \text{g} n_1 E_{51}^{3/2} M_{ej,\odot}^{-3/2} t_{\text{yr}}^3 \end{aligned} \tag{394}$$

There will be a time when $M_{ej} = M_{acc}$: the dynamics is now profoundly affected. When does this happen?

$$T_{ST,yr} = 150 \text{yr} E_{51}^{-1/2} M_{ej,\odot}^{5/6} n_1^{-1/3} \tag{395}$$

This is called **Sedov-Taylor time**. They studied these effects on nuclear explosions. So

[16]Recall that the magnetic field in Gaussian units (G) can be expressed in terms of cgs units as follows:

$$1\text{G} = 1\text{cm}^{-1/2}\text{g}^{1/2}\text{s}^{-1}$$

- When $t < T_{ST}$ we have the **ejecta dominated phase**. Here $v_{ej} \sim$const, and $R_{shock} \sim v_{ej}t$, i.e. the size of the shock increases linearly in time
- When $M_{acc} \gg M_{ej}$, then we cannot have additional energy $\frac{1}{2}M_{acc}v_{shock}^2 = E_{SN}$. The shock slows down, we have the so called **Sedov-Taylor phase**. So

$$E_{SN} = \frac{1}{2}\left(\rho_{ISM}v_{shock}4\pi R_{shock}^2 t\right)v_{shock}^2 \tag{396}$$

If we assume that $R_{shock} \sim t^{\alpha}$, since it cannot increase linearly always, otherwise the velocity would be constant and we have seen that it has to slow down:

$$v_{shock} \sim \frac{dR_{shock}}{dt} \sim t^{\alpha-1} \tag{397}$$

$$\implies \left(t^{\alpha-1}\right)^3 tt^{2\alpha} = \text{const} \implies t^{5\alpha-2} = \text{const} \tag{398}$$

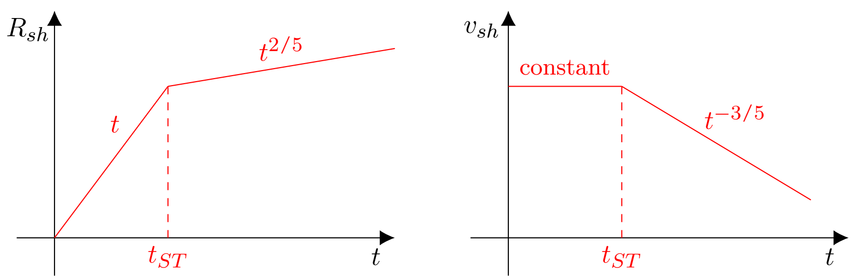
Therefore

$$\alpha = \frac{2}{5} \tag{399}$$

and we can write

$$\begin{aligned} R_{shock} &\propto t^{2/5} \\ v_{shock} &\propto t^{-3/5} \end{aligned} \tag{400}$$

So we have an ejecta dominated phase and a Sedov-Taylor phase.



Not all the supernova explode in the interstellar medium. A massive star that explodes as a core collapse supernova has already lost a significant part of its mass in stellar winds. So the explosion happens in the wind, not in vacuum.

$$\rho_{wind} = \frac{\dot{M}}{4\pi r^2 v_{wind}} \tag{401}$$

We can calculate the two phases also for the case of a star that explodes in its own wind.

Note also that core collapse supernovae are more frequent than supernovae IA. The typical size of a supernova at the Sedov-Taylor time is of order of few parsecs. During the Sedov-Taylor phase the velocity keeps dropping and at some point the material behind the shock becomes dense enough to start recombination (electrons and hydrogen start forming atoms). This is the end of the supernova life and it is called the **radiative phase**. In the radiative phase the shock velocity has slowed down to 100km/s. **All the phases last more or less** $20-30$ **thousand years** (explosion, Sedov-Taylor and radiative phases), taking standard parameters for the interstellar medium. Shock is dropping and particle acceleration follows all along.

So the question is: which sources (energetically speaking) feed the cosmic rays? These are indeed supernovae o at least something connected to them: star clusters or stellar winds or phases before an explosion (winds can be supersonic as well). These phenomena are ubiquitous.

ii.2 Acceleration time and maximum energy

Remember that our major issue is to accelerate particles to reach energies of interest, up to the PeV scale. Remember that the escape time from the galaxy is $\tau_{esc} = 90\text{Myr}\left(\frac{H}{3\text{kpc}}\right)$ at 10GeV of energy per nucleon. If the particles are feeling the perturbations of the magnetic field of the galaxy, $\tau_{esc} \sim \frac{H^2}{D}$.
Then we can write

$$D \simeq \frac{H^2}{\tau_{esc}} = \frac{\left[3\left(3.1 \times 10^{21}\right)\right]^2}{10^8\text{yr}} \sim 3 \times 10^{28}\frac{\text{cm}^2}{\text{s}} \tag{402}$$

always at 10GeV of energy per nucleon.
On the other hand

$$D = \frac{1}{3}r_L(p)v\frac{1}{\mathcal{F}} \overset{10\text{GeV/n}}{=} 3 \times 10^{28} \tag{403}$$

Therefore

$$\mathcal{F} = \frac{\frac{1}{3}10^{13}(3 \times 10^{10})}{3 \times 10^{28}} \sim 10^{-6} \tag{404}$$

where $r_L \sim 10^{13}$cm at the energy of 10GeV.
In order to satisfy the conditions derived from the spallation of boron, we need a perturbation of order of 10^{-6}. So a small perturbation leads to a gigantic effect. Here we are changing the time scales from H/c to a hundred million years by just adding this small perturbation!

Despite this we have also obtained that $\tau_{acc} = \frac{3}{u_1 - u_2}\left[\frac{D_1}{u_1} + \frac{D_2}{u_2}\right] \sim \frac{D}{u_1^2}$. If we use that $D \sim 3 \times 10^{28}\left(\frac{E}{10\text{GeV}}\right)^{\delta}\frac{\text{cm}^2}{\text{s}}$, close to the shock we obtain

$$u_1 \sim v_{shock} = 10000\frac{\text{km}}{\text{s}} \tag{405}$$

and the shock velocity can only get lower, so that the acceleration times can only get larger!
Therefore we can calculate the acceleration time for these reference values of the parameters:

$$\tau_{acc} = \frac{3 \times 10^{28}E_{10}^{\delta}}{10^{18}(\text{cm/s})^2} = 3 \times 10^{10}E_{10}^{\delta} = 10^3\text{yr}E_{10}^{\delta} \tag{406}$$

So if the shock is expanding in the interstellar medium, and if the diffusion coefficient is the one we just estimated, and given that the Sedov-Taylor phase begins just after 150 years, the energy is not increasing enough. We cannot possibly have this value for the diffusion coefficient close to the shock. Then we have to increase $\mathcal{F}$. **Why close to the shock we expect a lower value for the diffusion coefficient D?** The velocity is supersonic in the upstream and eventually it will become subsonic in the downstream. Therefore in the downstream it is easier to produce perturbations: the diffusion coefficient in the downstream can be very small. On the other hand, roughly $\tau_{acc} \sim D_1/u_1^2$, so reducing D_2 does not decrease the acceleration time (D_1 is still the same as that of the interstellar medium). But the shock is moving supersonically: we cannot send a message from the shock to the upstream! The only thing that can violate this criteria is the accelerating particles: they are the only one moving diffusively in the upstream. In the next section we'll see how this can happen.

V. Basics of non linear theory of CR acceleration

i. Magnetic field amplification

In the downstream we have a lot of things that can produce perturbations (a lot of turbolence). Even if D_2 is small, but D_1 does not change much, we obtain nothing. We need to work on D_1, but it is a mess. The shock wave moves supersonycally: we cannot send a message from the shock to the upstream. The only thing moving diffusively in the upstream are the accelerated particles. Therefore accelerated particle indeed do something! We need a non linear theory.

- We have already seen that applying momentum conservation in the upstream and adding the pressure of cosmic rays in the proximity of the shock, we develop a precursor, that means that the velocity of the fluid upstream is no longer constant spatially, it develops a profile. Therefore the compression factor is function of momentum: we no longer have a power law (previously it was solely dependent on the compression factor), even if these violations of the power law are not that dramatic.
- We also have the problem of acceleration time. Although we have shown that it is a first order acceleration process, it happens that for each cycle (upstream-downstream-upstream) particle energy increases only of $4/3v_{shock}/c$, which is $10^{-2} - 10^{-3}$ for typical parameters. Therefore the particle energy increases by only 0.1% every time it crosses the shock. The mechanism requires a lot of time to accelerate particles to a decent amount. We have also seen that assuming a value for the diffusion coefficient estimated from the B/C ratio, we obtain that the acceleration time, $\tau_{acc} \sim D/v_{shock}^2$, even for particles to be accelerated to 1GeV requires thousands of years, which is a significant time for a supernova remnant.

What are we missing? We can imagine that we shoot a beam of particles in the plasma. Because of the presence of Alfven waves the beam would start to diffuse and get broader until eventually the deflection can reach 90° and the particles start going backwards. We can think of this from the point of view of momentum conservation. We start with a situation where all the momentum is in one direction and then, at a certain point particles have momentum also in the opposite direction. Momentum has to be conserved. Where does the momentum go?
Our system is composed of Alfven waves, plasma and non thermal particles. These last ones start with a momentum in the z direction and end up having none. **The momentum is converted as perturbations of Alfven waves**. We can think of this as a particle being reflected by a mirror (like in Fermi's reasoning), but the mirror is not affected by the particle only if it has infinite mass, which is not the case. Momentum of the Alfven waves is only in the form of electric and magnetic fields, but since this phenomenum exists also in the reference frame of Alfven waves (where there is no electric field), it has to do with the magnetic field only. The momentum the particles are losing to deflect of a certain amount is somehow converted to perturbations of the magnetic field. This means that even if we are in a system with no Alfven waves, having a bunch of particles moving in the system is able to excite these waves.

We can also look at this from another point of view. We have a charged particle moving in a system with a magnetic field B_0. As a consequence it is going to produce its own electric and magnetic fields. If we go back to the Vlasov equation, we had electric and magnetic fields that were feeling the contributions from all the particles. We found the Alfven waves perturbing a little a system made of electrons and protons. If we add cosmic rays (that is a spectrum $\sim p^{-4}$), we still find Alfven waves (if the density of cosmic rays is minor that that of the gas, which is always the case), but this time the frequency develops a small imaginary part. If the imaginary part is negative, **the system is developing an instability**. This is exactly what happens. We have the real part which is still $\omega = v_A k$ plus an imaginary part.

What is the momentum transfer quantitatively? First notice that if the particles are already isotropic there cannot be a momentum transfer. We can have these effect as long as we have a small anisotropy. So let's take the distribution function and expand it for small anisotropies. We also introduce a drift velocity $\delta \sim \frac{v_{drift}}{v} \sim \frac{v_D}{c}$, which gives us the order of magnitude of the anisotropy, since the particles are streaming somewhat in a preferential direction.

$$f(p,\mu) = f_0(p)\left[1+\delta\mu\right] = f_0(p)\left[1+\frac{v_D}{c}\mu\right] \tag{407}$$

The drift velocity for the case of a shock wave moving in the interstellar medium is a few times the Alfven speed. The net momentum the particle carries is

$$\int_{-1}^{+1} d\mu 4\pi p^3 f_0(p) p\mu \left[1+\frac{v_D}{c}\mu\right] = \frac{2}{3}\underbrace{4\pi p^3 f_0}_{n(>p)} m_p c\gamma \frac{v_D}{c} \tag{408}$$

Remember that we are assuming that particles are relativistic. The momentum transfer stops when the particle distribution gets isotropic. When diffusing, particles are reducing the drift speed to the Alfven speed. The amount of momentum that is being lost by the particles by diffusing is:

$$\Delta P = \frac{2}{3} n(>p) m_p \gamma [v_D - v_A] \tag{409}$$

This is the change of momentum the particles are trying to experience. This happens at the timescale of 90° deflection

$$D(p) = \frac{1}{3} r_L v \frac{1}{\mathcal{F}(k)} \tag{410}$$

at $k = k_{res} = \frac{1}{r_L(p)}$.
If the function $\mathcal{F}$ is independent of k, i.e. the spectrum is scale invariant, then the diffusion coefficient reduces to $\frac{1}{3} r_L v$, which is the Bohm diffusion.
On the other hand, $D(p)$ can also be written, in general, as $D(p) = \frac{1}{3} v\lambda$ with the pathland $\lambda(p) = \frac{r_L(p)}{\mathcal{F}(k_{res})}$. As a consequence the time required to cover one pathland (which is also the time scale for 90° deflection), is:

$$\tau = \frac{\lambda(p)}{c} = \frac{r_L}{c\mathcal{F}} = \frac{pc}{eB_0}\frac{1}{c}\frac{1}{\mathcal{F}} = \frac{m_p c\gamma}{eB_0}\frac{1}{\mathcal{F}} = \Omega_c^{-1}\frac{\gamma}{\mathcal{F}} \tag{411}$$

where we see that more energetic particles require longer timescale for the deflection.
We have both the momentum loss and the time it takes to lose it:

$$\frac{\Delta P}{\tau} = \frac{2}{3} n(>p) m_p \not{\gamma} (v_D - v_A) \frac{\mathcal{F}}{\not{\gamma}} \Omega_c \tag{412}$$

The only place the momentum can go is to the Alfven waves, which gain a momentum of:

$$\left(\frac{\Delta P}{\tau}\right)_w = \frac{\delta B^2}{8\pi} \gamma_w \frac{1}{v_A} \tag{413}$$

where we have used dimensional arguments. In fact it has to be proportional to the energy density of the magnetic field ($\frac{\delta B^2}{8\pi}$). Then we need a quantity which is the inverse of a time, which we call γ_w, **the growth rate**, i.e. how fast the transfer of momentum happens (this is what we want to calculate!). Finally we need the inverse of a velocity, and the only velocity relevant to Alfven waves is the Alfven speed. At equilibrium the two variations have to be the same:

$$\begin{aligned}
\frac{2}{3} n(>p) m_p (v_D - v_A) \underbrace{\mathcal{F}}_{\frac{\delta B^2}{B_0^2}} \Omega_c &= \frac{\delta B^2}{8\pi} \gamma_w \frac{1}{v_A} \\
\frac{2}{3} n(>p) m_p (v_D - v_A) \Omega_c &= \frac{B_0^2}{8\pi} \gamma_w \frac{\sqrt{4\pi\rho}}{B_0} \\
\frac{2}{3} n(>p) m_p (v_D - v_A) \Omega_c &= \frac{1}{2} v_A m_p n_i \gamma_w
\end{aligned} \tag{414}$$

where in the last step we have multiplied and divided by $\sqrt{4\pi\rho}$. At this point, we can calculate γ_w:

$$\gamma_w = \frac{4}{3}\frac{n(>p)}{n_i}\frac{v_D - v_A}{v_A}\Omega_c \tag{415}$$

This is a growth rate, the rate at which the waves are modified. Basically it is the imaginary part of the frequency if we go to Fourier modes. The δB of the waves increases exponentially on a timescale given by $1/\gamma_w$. Let's apply it to the shock waves.

Let's apply this result to the shock. The transport equation is given by $u\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right]$. If we impose boundary conditions at the upstream infinity, we obtain that the current is constant. As a consequence $uf = D\frac{\partial f}{\partial z}$.
u is the velocity of the shock in the reference frame of the waves, that means that $u = v_{shock} - v_A$ and this is exactly $v_D - v_A$.

$$(v_{shock} - v_A)f = D\frac{\partial f}{\partial z} \tag{416}$$

Therefore:

$$\gamma_w = \frac{4}{3}\frac{n(>p)}{n_i}\frac{v_D - v_A}{v_A}\Omega_c = \frac{4}{3}\frac{1}{n_i}\frac{1}{v_A}\Omega_c D\frac{\partial n(>p)}{\partial z} \tag{417}$$

Let's use this new form of γ_w:

$$\gamma_w = \frac{4}{3}\frac{1}{n_i}\frac{1}{v_A}\Omega_c\frac{1}{3}\frac{r_L c}{\mathcal{F}}\frac{\partial n(>p)}{\partial z} \tag{418}$$

What is the equation that describes how δB increases with time? We want to write the growth rate as $\frac{D\mathcal{F}}{Dt} = \gamma_w\mathcal{F}$, which is what a growth rate is. If we assume the system to be globally stationary:

$$\cancel{\frac{\partial \mathcal{F}}{\partial t}} + v_{shock}\frac{\partial \mathcal{F}}{\partial z} = \gamma_w\mathcal{F} \tag{419}$$

We are non considering here the precursor effect, so we can write:

$$\frac{\partial \mathcal{F}}{\partial z} = \frac{1}{v_{shock}}\frac{r_L c}{\mathcal{F}}\frac{\partial n(>p)}{\partial z}\mathcal{F}\frac{4}{9}\frac{1}{n_i}\frac{1}{v_A}\Omega_c \tag{420}$$

All these derivatives are zero at upstream infinity, so we obtain what happens at the shock location:

$$\begin{aligned}\implies \mathcal{F} &= \frac{4}{9}\frac{1}{v_{shock}}\frac{m_p c^2}{eB_0}\gamma\frac{1}{n_i}\frac{B_0}{\sqrt{4\pi\rho}}\frac{eB_0}{m_p c}n(>p) \\ &= \frac{4}{9}\frac{c}{v_{shock}}\gamma\frac{1}{n_i}n(>p)\frac{1}{v_A}\end{aligned} \tag{421}$$

γ is basically the equivalent of p: $\gamma \sim p$:

$$n(>p) = 4\pi p^3 f, \qquad f \sim p^{-4} \quad \implies \quad n(>p) \sim p^{-1} \tag{422}$$

$\mathcal{F}$ **is scale invariant**, if we multiply all terms by m_p:

$$\delta B^2 = \frac{4}{9}\frac{m_p c}{m_p v_{shock}}\frac{1}{n_i}B_0^2\frac{B_0}{\sqrt{4\pi\rho}} \tag{423}$$

The total energy of cosmic rays is

$$\begin{aligned}\epsilon_{CR} &= \int dp 4\pi p^2 pcf \\ &= 4\pi c A(mc)^4\int_1^{p_{max}/mc} dx x^{-1} \\ &= 4\pi c A(mc)^4 \ln\left(\frac{p_{max}}{mc}\right)\end{aligned} \tag{424}$$

where we have used diffusive shock approximation $f(p) = A\left(\frac{p}{p_0}\right)^{-4}$ and have assumed that the particles are relativistic $p_0 \sim mc$.
ϵ_{CR} is the energy in terms of cosmic rays, we can take from the kinetic energy of plasma. Let's normalize it to it

$$\epsilon_{CR} = \xi_{CR}\rho v_{shock}^2 \tag{425}$$

Thanks to the normalization we can calculate A:

$$4\pi c A (m_p c)^4 \ln\left(\frac{p_{max}}{mc}\right) = \xi_{CR} m_p n_i v_{shock}^2 \tag{426}$$

$$\implies A = \xi_{CR}\frac{m_p n_i v_{shock}^2}{4\pi c (m_p c)^4 \ln\left(\frac{p_{max}}{mc}\right)} \tag{427}$$

We managed to connect the acceleration efficiency to the number density:

$$n(>p) = 4\pi p^3 \xi_{CR}\frac{m_p n_i v_{shock}^2}{4\pi c (m_p c)^4 \ln\left(\frac{p_{max}}{mc}\right)}\left(\frac{p}{m_p c}\right)^{-4} = \frac{\xi_{CR}}{m_p c}\frac{m_p n_i v_{shock}^2}{c \ln\left(\frac{p_{max}}{m_p c}\right)}\left(\frac{p}{m_p c}\right)^{-1} \tag{428}$$

We can write the normalized distribution function as:

$$n(>p) = \xi_{CR}\frac{\rho v_{shock}^2}{c m_p c \Lambda}\left(\frac{p}{m_p c}\right)^{-1} \tag{429}$$

with

$$\Lambda = \int_1^{\frac{p_{max}}{m_p c}} \frac{dx}{x} = \ln\left(\frac{p_{max}}{m_p c}\right) \tag{430}$$

$$\implies \mathcal{F} = \frac{4}{9}\frac{1}{\rho}\frac{1}{v_A v_{shock}} p \frac{\xi_{CR}\rho v_{shock}^2}{m_p c \Lambda}\frac{m_p c}{p} \tag{431}$$

The momentum goes exactly away because we have assumed strong shock (otherwise we would have had a weak dependence on p). Finally we obtain:

$$\mathcal{F} = \frac{4}{9}\frac{v_{shock}}{v_A}\xi_{CR}\frac{1}{\Lambda} \tag{432}$$

where $\frac{v_{shock}}{v_A}$ is the alfvenic Mach number.
This is a scale invariant spectrum: particles accelerating with a strong shock induce scale invariant perturbations. In terms of the diffusion coefficient $D = \frac{1}{3} r_L c \frac{1}{\mathcal{F}}$, we have no dependence on k (otherwise we should have chosen the resonant wave number). This implies that the diffusion coefficient is always linear in momentum (the momentum dependence of the Larmor radius). So we have a Bohm diffusion coefficient. Moreover:

$$\mathcal{F} \equiv \frac{M_A \xi_{CR}}{\Lambda} \tag{433}$$

where at minimum $M_A \sim 100$ and the acceleration efficiency for a strong shock is of order $\xi_{CR} \sim 0.1$. The result has changed by ~ 7 orders of magnitude, it's the key point of non linearity (the particles are diffusing and they are producing themselves the perturbations on which they are diffusing). What we have obtained is perturbations are of order of B_0. And when δB becomes large we cannot do analytic calculations. Remember though that when we calculated the particle's trajectory, we obtained that it spirals unperturbed unless it achieves a resonance, i.e. $k = \frac{1}{r_L}$. Then the particle's trajectory changes. The resonance was calculated using the Larmor radius in the field B_0, but now δB becomes even larger than B_0. So what is the Larmor radius? It doesn't make sense to make these calculations analytically (we should go to second order perturbations and the difficulty is unbearable).
The best way to proceed is to solve the Vlasov equation numerically. We solve for the motion of each individual particle in the system, which feels the magnetic and electric fields of all the other particles.

We can think of this process as an absorption (diffusion) and production (self generation) of Alfven waves. There is indeed a formalism similar to Feynman diagrams to describe this process.

We can ask again what is the acceleration time of the shock:

$$\tau_{acc} = \frac{D}{v_{shock}^2} = \frac{1}{3} r_L c \frac{\Lambda}{M_A \xi_{CR}} \frac{1}{v_{shock}^2} = \frac{3 \times 10^{22} \mathrm{E_{GeV}}}{v_{shock}^2} = \frac{3 \times 10^{22} \mathrm{E_{GeV}}}{10^{16}} \sim \text{hours} \tag{434}$$

Without these non linear processes, all the theory of diffusive shock acceleration would be useless. We would not be able to reach meaningful energies. What happens here is the following process. If particles are trying to stream away with a velocity greater than v_A, they will induce instabilities, that will inhibit that from happening.

i.1 Vlasov description and development of streaming instabilities

There is a more formal way to obtain the above results. This is a general approach. Let's turn back to the plasma system made of electrons and ions, interacting under the magnetic and electric fields they produce themselves plus a B_0 magnetic field. The electric and magnetic fields are connected to the distribution function through the sources. We perturb our equations (Vlasov for each species plus Maxwell) and we obtain what kind of perturbations are allowed in the system. Recall that we have obtained:

$$\frac{k^2 c^2}{\omega^2} = 1 + \sum_\alpha \frac{4\pi^2 q_\alpha^2}{\omega} \int dp \int d\mu \frac{p^2 v (1-\mu^2)}{\omega - kv\mu \pm \Omega_\alpha} \left\{ \frac{\partial f_\alpha}{\partial p} + \frac{1}{p} \frac{\partial f_\alpha}{\partial \mu} \left(\frac{kv}{\omega} - \mu \right) \right\} \tag{435}$$

We have solved it for a cold plasma and have obtained the Alfven waves. To derive the previous result in a formal way, we add the cosmic rays: so we have $\alpha = e^-$, $\alpha =$ ions and $\alpha =$ cosmic rays. Specializing to the case of the shock waves: we are sitting in the reference frame of the shock. In this frame electrons and protons in the upstream are coming towards us. So the distribution function of electrons and protons[17] (we are still assuming a cold plasma):

$$f_{e,p}(p,\mu) = \frac{A}{4\pi p^2} \delta(p - n_{e,p} v_{e,p}) \delta(\mu - 1) \tag{436}$$

A is the normalization constant that can be calculated imposing:

$$\int_{-1}^{+1} d\mu \int_0^{+\infty} 4\pi p^2 \frac{A}{4\pi p^2} \delta() \delta() = n_{e,p} \tag{437}$$

$$\implies f_{e,p} = \frac{n_{e,p}}{2\pi p^2} \delta(p - m_{e,p} v_{e,p}) \delta(\mu - 1) \tag{438}$$

We need something similar for the accelerating particles as well (cosmic rays):

$$f_{CR}(p) = A_{CR} \left(\frac{p}{p_0} \right)^{-4} \implies f_{CR}(p) = \frac{n_{CR}}{4\pi p_0^3} \left(\frac{p}{p_0} \right)^{-4} \tag{439}$$

where A_{CR} is the normalization to the density of cosmic rays.
In general, a system doesn't like spare charges. If a shock comes in, we have positive charges (cosmic rays) which accumulate within a diffusion length: **plasma tries to retain charge neutrality**

$$n_e = n_p + n_{CR} \tag{440}$$

Furthermore, it doesn't like to have currents. In the shock frame the charge current of electrons and protons need to be the same:

$$n_e v_e = n_p v_p \tag{441}$$

[17] Before we didn't have the dependence on μ because we were sitting in the plasma reference frame

If there are no accelerating particles then $v_e = v_p$, but if there are accelerating particles, then:

$$v_e = \frac{n_p}{n_e} v_p = \frac{n_p}{n_p + n_{CR}} v_p \sim \left(1 - \frac{n_{CR}}{n_p}\right) v_p \tag{442}$$

where we have assumed $\frac{n_{CR}}{n_p} \ll 1$. We have obtained that v_e has become smaller to compensate for the fact that we have added accelerated particles to the system. We can now write the dispersion relation as

$$\frac{k^2 c^2}{\omega^2} = 1 + \chi_e + \chi_p + \chi_{CR} \tag{443}$$

If we repeat the exercise taking into account this complication, we'll obtain

$$\chi_e + \chi_p = \left(\frac{c}{v_A}\right)^2 \frac{\tilde{\omega}^2}{\omega^2} \pm \left(\frac{c}{v_A}\right)^2 \frac{\Omega_p}{\omega} \frac{n_{CR}}{n_p} \tag{444}$$

where $\tilde{\omega} = \omega - kv_{shock}$ is the Doppler shifted frequency (comoving frequency). We are not in the plasma reference frame as before. If $n_{CR} = 0$ we get the Doppler shifted Alfven waves, what we have already obtained (now our waves instead of moving with v_A they are moving with velocity $v_A + v_{shock}$).
We need to calculate χ_{CR}:

$$\chi_{CR} = \frac{4\pi e^2}{\omega} \int dp \int d\mu \frac{p^2 v(1-\mu^2)}{\omega + kv\mu \pm \Omega_{CR}} \frac{n_{CR}}{4\pi p_0^3} \frac{\partial}{\partial p} \left[\left(\frac{p}{p_0}\right)^{-4} \Theta(p - p_0)\right] \tag{445}$$

The integral in $d\mu$ causes trouble because it has a pole, but it can be solved using

$$\omega = \omega_R + i\omega_I, \quad \text{with} \quad \omega_I \ll \omega_R \tag{446}$$

and writing it as the Cauchy principal value of the integral (the principal part is where we take away the ω):

$$\begin{aligned} I &= \int_{-1}^{+1} d\mu \frac{1-\mu^2}{\omega + kv\mu \pm \Omega_{CR}} \\ &= \mathcal{P} \int d\mu \frac{1-\mu^2}{kv\mu \pm \Omega_{CR}} - i\pi \int_{-1}^{+1} d\mu (1-\mu^2) \delta(kv\mu \pm \Omega_{CR}) \end{aligned} \tag{447}$$

where the imaginary expression is the one that causes instabilities and expresses the resonance condition.
The same process makes the particles diffuse and creates Alfven waves. The perturbations allowed to propagate have

$$\omega_R = \text{Re}[\omega] \simeq kv_A \tag{448}$$

and

$$\omega_I = \text{Im}[\omega] \simeq \frac{\pi}{8} \frac{n_{CR}(p > p_{res})}{n_p} \Omega_{CR} \frac{v_{shock} - v_A}{v_A} \tag{449}$$

This is similar to what we have found from the conservation of momentum. It is proper way to calculate the growth rate of these instabilities. The Fourier mode has $\exp[-i\omega t + ikz]$, it has developed an imaginary part. So we have instabilities.
If we don't assume $\omega_I \ll \omega_R$, we get another instability, called a non resonant hybrid instability which grows much much faster under certain conditions.

Some numbers

Let's look at some numbers:

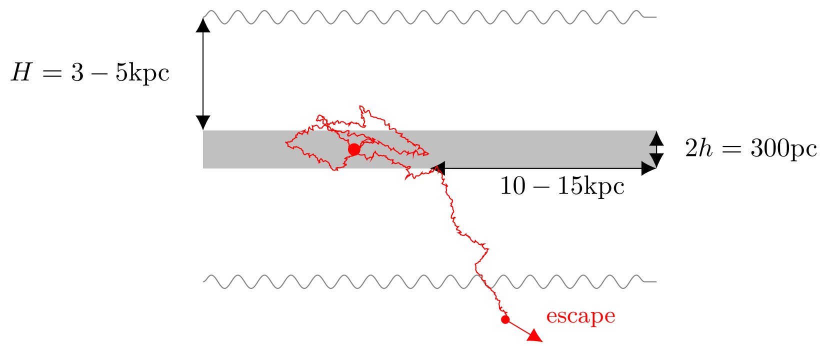
- We said that the number density of the ISM is roughly 1cm^{-3}
- The energy density of the ISM is more or less 1 eV per cubic centimeter, which is also the energy density of the electromagnetic field and that of cosmic rays
- Taking an average cosmic ray particle to have an energy of 1GeV, we obtain a number density for cosmic rays which is of order 10^{-9} cosmic rays per cubic centimeter.

From the point of view of densities, cosmic rays are definitely not important. Nevertheless, when using the perturbed Vlasov equation to describe a system made of plasma of electrons and protons and this tiny tiny fraction of cosmic rays, we end up with a dispersion relation, which admit imaginary ω solution. Since Magnetic field perturbations can be written as $\delta\vec{B} = \delta B \exp\left[-i\omega t + \vec{k}\cdot\vec{z}\right]$, depending on the sign of the imaginary part of ω we can obtain exponentially growing instabilities!

VI. CR transport in the galaxy

I. CR spectrum from inside the Galaxy

We specialize to our own galaxy because we have data.



This is a toy model of our galaxy: $n_d \gg n_H$. All the sources are in the **disk**. Then we have this extended region, the **halo**, of magnetized gas. We know that it is magnetized because we observe synchrotron emission. The disk is really thin compare to the halo: it resembles a shock surface. Moreover, **here we are assuming that the background plasma is not moving**.
We want to describe the following problem: assuming there are sources of non thermal particles (cosmic rays) in the disk of the galaxy, what do we see? From Earth we see a steeper power law, $E^{-2.7}$. What is the connection between the cosmic rays in the Galaxy and what we observe on Earth? Bear in mind that because of diffusion we cannot retrace the position of the source. So the particles we are interested in are produced somewhere in the galaxy, then they diffuse and eventually they reach the boundary of the Galaxy and finally escape. Let's use our transport equation:

$$\cancel{\frac{\partial f}{\partial t}} + \cancel{u\frac{\partial f}{\partial z}} - \cancel{\frac{1}{3}\frac{du}{dz}p\frac{\partial f}{\partial p}} = \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] \tag{450}$$

where z is now the vertical distance. Our Galaxy is about several billion years old and the observation of light elements suggest that they spend at most a hundred million years in our Galaxy. So the dynamical scale of the Galaxy is much longer than the phenomenon we are dealing with: **we are authorized to assume stationarity**. Moreover, there is no motion of the interstellar medium. The equation becomes very simple:

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = -Q \tag{451}$$

- f is the distribution function of cosmic rays at any location in the galaxy and at any momentum p
- Q is the injection term: $Q(z,p) = q_0(p)\delta(z)$, because the disk is very thin.

Eventually cosmic rays will leave the system. The boundary condition is $f(z = \pm H) = 0$ (free escape boundary condition). Therefore:

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = -q_0(p)\delta(z) \tag{452}$$

In the case of supernova $q_0(p) = \frac{N(p)\mathcal{R}_{SN}}{\pi R_d^2}$, with R_d the radius of the disk, $\mathcal{R}_{SN}$ the rate of supernovae in the galaxy and $N(p)$ the spectrum of one supernova. We are also assuming that **the diffusion coefficient has only a dependence on momentum**.
If $z \neq 0$:

$$D\frac{\partial f}{\partial z} = k(z) \implies \frac{\partial f}{\partial z} = \frac{k}{D} \tag{453}$$

At $z = 0$ we have $f(p, z) = A + Bz \implies f = f_0(p)$, the distribution of cosmic rays **in the disk**.
At $z = H$ we have $f = 0$.
Therefore for $z > 0$ we obtain:

$$f = f_0(p)\left[1 - \frac{z}{H}\right] \tag{454}$$

The distribution function in the halo decreases linearly, and goes to zero at the edge, mimicking the free escape.
Now we integrate in a small neighborhood of the disk:

$$D\frac{\partial f}{\partial z}\bigg|_{0^+} - D\frac{\partial f}{\partial z}\bigg|_{0^-} = q_0(p) \implies 2D\frac{\partial f}{\partial z}\bigg|_{0^+} = -q_0(p) \tag{455}$$

Using (454) we obtain:

$$\frac{\partial f}{\partial z}\bigg|_{0^+} = -\frac{f_0}{H} \tag{456}$$

So we can write, since the system is symmetric:

$$2D\frac{f_0}{H} = q_0(p) \implies f_0(p) = \frac{q_0(p)H}{2D} \tag{457}$$

We have obtained the distribution function in the disk. In the case of supernovae:

$$f_0(p) = \frac{N(p)\mathcal{R}_{SN}}{2\pi R_d^2 H}\frac{H^2}{D} \tag{458}$$

where $N(p)\mathcal{R}_{SN}$ is the number of particles injected per unit time by all supernovae, H^2/D is the escape time, $2\pi R_d^2 H$ is the total volume of the galaxy.
The spectrum now has a correction in terms of the momentum dependence due to the diffusion coefficient. Because the diffusion coefficient is always a growing function of momentum, the spectrum we see sitting in the Galaxy is always steeper than the spectrum we produce in the sources. If the source produces a spectrum of the type E^{-2}, we are going to see $E^{-2-\delta}$, where δ is the energy dependence of the diffusion coefficient. **Diffusion in the Galaxy changes the momentum dependence if we are sitting inside the Galaxy**. The spectrum at the Earth is different from the one produced at the accelerator. Therefore, if we do not know D (the microphysics of the transport) we are in trouble.

Let's make some remarks on the assumptions we have made so far. We have assumed free escape boundary conditions, so that

$$\begin{cases} z \neq 0 \implies \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = 0 \\ z = \pm H \implies f(z) = 0 \end{cases} \tag{459}$$

We have derived $f(z, p) = f_0(p)\left[1 - \frac{z}{H}\right]$ and therefore $D\frac{\partial f}{\partial z} = -\frac{f_0}{D}$. We have obtained a constant expression, non depending on the place where we are. Even at the boundary, where the particles are escaping, it is non zero. Physically $D\frac{\partial f}{\partial z}$ is the **diffusion current**, it is the flux of the particles escaping the system.

i.1 CR spectrum from outside the Galaxy

We have just seen that the spectrum we see on Earth is different from the one produced at the shock. There is an interesting question. What kind of spectrum would we see from Andromeda? Thanks to the **Liouville theorem** (stationarity), if we are injecting a given number of particles, we see that same number of particles, otherwise we could not be in a stationary regime. We would see the real spectrum!
There is also a formal way to arrive at the same conclusion. We have said that the flux is:

$$D\frac{\partial f}{\partial z} = -D\frac{f_0}{H} = -\frac{D}{H}\frac{N(p)\mathcal{R}_{SN}}{2\pi R_d^2}\frac{H}{D} = -\frac{N(p)\mathcal{R}_{SN}}{2\pi R_d^2} \tag{460}$$

So the total number of particles leaving the system is

$$\mathcal{N}(p) = 2\pi R_d^2 \frac{N(p)\mathcal{R}_{SN}}{2\pi R_d^2} \equiv N(p)\mathcal{R}_{SN} \tag{461}$$

Sitting inside the source we are forced to probe the diffusion coefficient: **we need a way to split the source from the diffusion coefficient** since we observe a combination of the two.

i.2 Physical meaning of free escape boundary condition

We also said that $f(z = \pm H) = 0$, but **what does it mean physically that the distribution function is going to zero?** What is the meaning of the free escape boundary condition. If we are sitting near the edge of the Galaxy, the particles are moving nearly freely. Be f_* the distribution function immediately outside the halo of the galaxy, the escaping flux is:

$$\phi_{esc} = D\frac{\partial f}{\partial z} = D\frac{f_0}{H} \tag{462}$$

But the flux needs to be conserved in order to avoid accumulation of particles anywhere! Therefore:

$$\phi_{esc} \equiv f_* \cdot c \tag{463}$$

Remember that outside the particles are not diffusing, but streaming at velocity c, they are ballistic. So the distribution is not exactly zero.
On the other hand, we can always write $D = \frac{1}{3}c\lambda_d$. Therefore

$$\frac{1}{3}c\lambda_d\frac{f_0}{H} = f_*c \implies f_* = f_0\left(\frac{\lambda_d}{H}\right) \tag{464}$$

so f_* is small.
Moreover, $D = \frac{1}{3}r_L c\frac{1}{\mathcal{F}}$, so what we are requiring is that:

$$\lambda_d = \frac{r_L}{\mathcal{F}} \ll H \tag{465}$$

This condition is fulfilled at almost all energies (not when the pathland becomes of the order of the size of the galaxy, which takes place at energies of order of $10^{17} - 10^{18}$eV). Therefore when we are using the free escape boundary condition, we are assuming that the pathland is sufficiently small with respect to H.

II. Transport of stable nuclei

We have seen that the cosmic rays we observe on Earth are atomic nuclei, mainly protons. The main channel of energy loss for protons is inelastic *pp* scattering with pion production, with a typical cross section of order $\sigma_{pp} \sim 3 \times 10^{-26}\text{cm}^2$.
On the other hand, the time scale associated with the energy loss associated to this process is:

$$\tau_{loss} = \frac{1}{\bar{n}\sigma_{pp}c} \tag{466}$$

If we evaluate it, we see that it is large enough that we can ignore energy losses for protons.
Unfortunately, this is not the case for heavier nuclei, like the carbon. The heavier the nuclei, the main channel of losses is through spallation of these nuclei. Therefore the transport equation for a primary nucleus needs to be changed to reflect this fact:

$$-\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = q_0(p)\delta(z) - \frac{f}{\tau_{sp}} \tag{467}$$

We have a minus sign because we are loosing particles because f is being spallated. f is divided by the rate of distruction of that given nucleus (we are assuming that all the gas is in the disk of the Galaxy):

$$\frac{1}{\tau_{sp}} = 2hn_d\sigma_{sp}\delta(z)c \tag{468}$$

where $2h$ is the thickness of the disk.

$$\implies -\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = q_0(p)\delta(p) - f2hn_d\sigma_{sp}c\delta(z) \tag{469}$$

The loss term is analogue to the injected term. So mathematically speaking, if $z \neq 0$:

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = 0 \implies \frac{\partial f}{\partial z} = \text{const} \implies f = f_0\left[1 - \frac{z}{H}\right] \tag{470}$$

What has changed is f_0. How can we calculate it? Let's integrate around $z = 0$:

$$\begin{aligned} -D\frac{\partial f}{\partial z}\bigg|_{0^+} + D\frac{\partial f}{\partial z}\bigg|_{0^-} &= q_0(p) - 2hn_dc\sigma_{sp}f_0 \\ -2D\frac{\partial f}{\partial z}\bigg|_{0^+} &= q_0(p) - 2hn_dc\sigma_{sp}f_0 \\ 2D\frac{f_0}{H} + 2hn_dc\sigma_{sp}f_0 &= q_0(p) \\ f_0(p) &= \frac{q_0(p)}{2\frac{D}{H} + 2hn_d\sigma_{sp}c} \\ f_0(p) &= \frac{q_0}{2D}H\frac{1}{1 + n_d\frac{h}{H}c\sigma_{sp}\frac{H^2}{D}} \end{aligned} \tag{471}$$

Then we have

$$n_d\frac{h}{H}\sigma_{sp}c\frac{H^2}{D} = \frac{\tau_{esc}}{\tau_{sp}} \implies \tau_{sp}^{-1} = n_d\frac{h}{H}\sigma_{sp}c \tag{472}$$

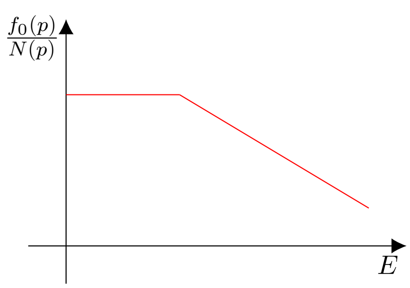
Observe that now the spallation time scale has a mean value for the density: it is as if particles diffusing in the Galaxy are feeling the mean value of the density.
Therefore considering the case of supernovae:

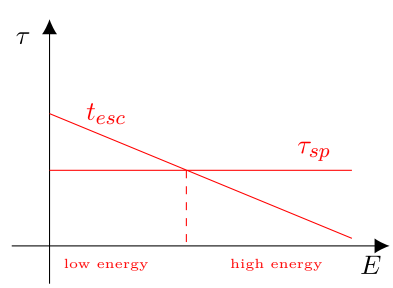
$$f_{0,SN}(p) = \frac{N(p)\mathcal{R}_{SN}}{2\pi R_d^2 H}\frac{H^2}{D}\frac{1}{1 + \frac{\tau_{esc}}{\tau_{sp}}} \tag{473}$$

- If $\tau_{esc} \ll \tau_{sp}$: the spallation is irrelevant (it escapes before it can spallate) and we obtain the standard result
- If $\tau_{sp} \ll \tau_{esc}$: we have that

$$f_{0,SN} \sim N(p)\tau_{esc}\frac{\tau_{sp}}{\tau_{esc}} \sim N(E)\tau_{sp} \tag{474}$$

The spallation is dominant. **The effect of spallation is that to reproduce the source spectrum again**. We are assuming that τ_{sp} does not depend on energy. This is a good assumption, because nuclear cross sections depend very weakly on energy.





At low energies we expect to reproduce the source spectrum!

Let's start again with the transport equation (we are always assuming that space is 1d)[18] and repeat the same calculations we have just done with the objective of rewriting the solution in terms of a quantity called **grammage**.

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial z} - \frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial f}{\partial p} = \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] + Q \tag{475}$$

Remember that we are also using a simple model for the galaxy, where all the sources are in the disk of the galaxy. Assuming stationarity:

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = -Q \equiv q_0(p)\delta(z) \tag{476}$$

We have found that the solution is:

$$f(z,p) = f_0(p)\left[1 - \frac{|z|}{H}\right] \tag{477}$$

where, imposing the free escape boundary condition, we obtained:

$$f_0(p) = q_0(p)\frac{H}{D} = \frac{N(p)\mathcal{R}}{2\pi R_d^2 H}\frac{H^2}{D} \tag{478}$$

The observed spectrum is steeper than the source spectrum exactly by the momentum dependence of the diffusion coefficient:

$$D(p) \sim p^{\delta}, \qquad N(p) \sim p^{-4} \qquad \Longrightarrow\ f_0(p) \sim p^{-4-\delta} \tag{479}$$

We have done the same exercise also for the transport of nuclei. The equation is basically the same:

$$-\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = q_0(p)\delta(z) - \frac{f}{\tau_{sp}(z)} \tag{480}$$

where the spallation time has a z dependence because particles do spallation only in the disk. Since the disk is small we use a δ function but we multiply it by the thickness of the disk $2h$:

$$\frac{f}{\tau(z)} = f n_d \sigma_{sp} v 2h\delta(z) \tag{481}$$

Now we see that all terms are proportional to δ-functions. f is the function we want to calculate:

$$-\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = \frac{N\mathcal{R}}{\pi R_d^2}\delta(z) - 2hn_d v\sigma_{sp} f\delta(z) \tag{482}$$

For $z \neq 0$ the RHS is identically zero and the spatial solution is identical to the one we obtained before:

$$f = f_0(p)\left[1 - \frac{|z|}{H}\right] \tag{483}$$

where we adopted the usual free escape boundary conditions at the edge of the halo.
What changes is $f_0(p)$, the integration gives us:

$$-2D\frac{\partial f}{\partial z}\bigg|_{0^+} = \frac{N\mathcal{R}}{\pi R_d^2} - 2hn_d v\sigma_{sp} f_0(p) \tag{484}$$

Therefore:

$$f_0\left[1 - \frac{X(p)}{X_{cr}}\right] = \frac{N\mathcal{R}}{2\pi R_d^2}\frac{H^2}{D} \tag{485}$$

[18] Actually we have turbulences (**random walk of magnetic field lines**), we also have the **perpendicular diffusion** phenomenon, which is studied only using simulations. Here we only study the motion forward and backward along the z axis.

where $X(p)$ is the **grammage**, i.e. how much material we cross per unit surface.
We have to solve:

$$2\frac{D}{H}f_0 = \frac{N\mathcal{R}}{\pi R_d^2} - 2hn_d v\sigma_{sp} f_0 \tag{486}$$

Therefore:

$$\begin{aligned} f_0\left[\frac{2D}{H} + 2hn_d\sigma_{sp}v\right] &= \frac{N\mathcal{R}}{\pi R_d^2} \\ f_0\left[1 + n_d\frac{h}{H}v\sigma_{sp}\frac{H^2}{D}\frac{m_p}{m_p}\right] &= \frac{N\mathcal{R}}{2\pi R_d^2 H}\frac{H^2}{D} \end{aligned} \tag{487}$$

We need a mass over surface to define a grammage, that's why we multiply and divide by the proton mass:

$$X(p) = n_d\frac{h}{H}v\frac{H^2}{D}m_p \equiv \frac{\mathrm{g}}{\mathrm{cm}^2} \tag{488}$$

So while moving across the Galaxy particles traverse a density on average $n_d\frac{h}{H}$ for a time that it takes particles to leave the Galaxy $\frac{H^2}{D}$.
The critical grammage is:

$$X_{cr} = \frac{m_p}{\sigma_{sp}} \tag{489}$$

Critical grammage basically means that in order for spallation to be important the grammage should be at least X_{cr}. If the grammage is larger than the critical one then the spallation is very effective. If it is much smaller than the critical grammage the spallation is not very effective. Therefore we can rewrite the above equation:

$$f_0\left[1 + \frac{X(p)}{X_{cr}}\right] = \underbrace{\frac{N\mathcal{R}}{2\pi R_d^2 H}}_{\text{injection rate}}\frac{H^2}{D} \tag{490}$$

Note that the injection rate is in the whole Galaxy.
The stationarity assumption means that particles are produced at the same rate that they are escaping.

- If $X(p) < X_{cr}$ then we obtain the same result as there were no spallation.
- If $X(p) > X_{cr}$, spallation is important:

$$\begin{aligned} f_0(p) &= \frac{N\mathcal{R}}{2\pi R_d^2 H}\frac{H^2}{D}\frac{X_{cr}}{X(p)} \\ &= \frac{N\mathcal{R}}{2\pi R_d^2 H}\frac{H^2}{D}\frac{m_p}{\sigma_{sp}}\frac{1}{n_d\frac{h}{H}vm_p\frac{H^2}{D}} \\ &= \frac{N\mathcal{R}}{2\pi R_d^2 H}\frac{1}{n_d\frac{h}{H}v\sigma_{sp}} \\ &= \frac{N\mathcal{R}}{2\pi R_d^2 H}\tau_{sp} \end{aligned} \tag{491}$$

  where τ_{sp} is the spallation time of the whole galaxy.

Remember that we always have the injection rate per unit volume times the relevant timescale.
When spallation is important? The spallation cross section does not depend on energy (it is very weakly dependent), it is constant (the heavier the nucleus the smaller the spallation time). At sufficiently low energy spallation is important (because τ_{esc} scales with the diffusion coefficient and therefore becomes smaller and smaller with increasing energies) and the spectrum we see is the same as the source spectrum. That is what we observe for heavy nuclei (heavier than hydrogen): moving towards lower energies the spectra become harder and harder.

The main limitations in these calculations come from the fact that the cross sections for these processes are poorly known, like the cross section for spallation of carbon into boron.

Usually the diffusion coefficient is considered as a free parameter, because it is extremely hard to measure the power spectrum of modes of the perturbation alike in our own Galaxy and outside of our Galaxy (like in an active galactic nucleus). **Is there some way to measure directly the diffusion coefficient?**

III. Transport of stable secondary nuclei: the B/C ratio

We have seen that the microphysics is contained in the diffusion coefficient, which in papers is often treated as a free parameter. **Is it possible to measure the diffusion coefficient?** The diffusion coefficient always appears in combinations like H/D. There is never D or H alone. The problem is that we do not know the extent of the halo. Therefore we do not know is the diffusion coefficient as well. How can we disentangle H/D?
We'll study the following phenomenon: there are some special nuclei, **borum**, **lithium** and **berillium** that are practically absent in the ISM. These are rare, they are not produced during the Big Bang nucleosynthesis (as hydrogen and helium). After three minutes after the Big Bang the temperatures are already too low to produce heavier elements. After about a billion years after the Big Bang we start forming stars, but in the stars they are destroyed faster than they are produced. Fortunately stars make a lot of carbon, oxygen, iron, etc...
But shock accelerates the particles that already are in the ISM. The particles the supernova is ejecting are way behind the shock, they are the reason the shock exists, but what is being accelerated are the particles already present in the ISM. So why do we observe borum, berillium and lithium in the cosmic rays, if they are not present in the ISM? They are the result of a spallation process.
Let's consider carbon:

$$f_C = \frac{q_C(p)}{2n_d h m_p v} \frac{1}{\frac{1}{X(p)} + \frac{1}{X_{cr}}} \tag{492}$$

Carbon is described by f_C and it's producing boron through spallation: the energy per nucleon is conserved. Let's analyze the injection rate per unit time of boron:

$$4\pi p^2 Q_B(p)dp = f_C(p')dp' 2hn_d v\sigma_{sp}\delta(z)4\pi p'^2 \tag{493}$$

We have:

- The momentum of carbon: $p' = Am_p c\beta\gamma$
- The momentum of boron: $p = (A-1)m_p c\beta\gamma$

Therefore:

$$\implies \frac{dp'}{A} = \frac{dp}{A-1} \implies \frac{dp'}{dp} = \frac{A}{A-1} \tag{494}$$

and

$$Q_B(p) = \left[f_C\left(\frac{A}{A-1}p\right)\right]\left(\frac{A}{A-1}\right)^3 2hn_d v\sigma_{sp}\delta(z) \equiv q_B\delta(z) \tag{495}$$

The only thing that has changed is the injection rate which is now the spallation production of boron. Therefore:

$$f_B = \frac{q_B}{2n_d h v m_p} \frac{1}{\frac{1}{X(p)} + \frac{1}{X_{cr,boron}}} \tag{496}$$

but $q_B \propto f_C$, in other words the rate of injection of boron is proportional to the equilibrium solution of carbon:

$$\implies \frac{f_B}{f_C} \propto \frac{1}{\frac{1}{X(p)} + \frac{1}{X_{cr,boron}}} \tag{497}$$

There is no more dependence on pretty anything, but on grammages. Let's do the usual limits:

- If the grammage is small compared to X_{cr}, we have spallation but it's weak: $\frac{f_B}{f_C} \propto X(p)$
- If the grammage is large, $X(p) \gg X_{cr}$: $\frac{f_B}{f_C} \to$ constant

What is exactly $X(p)$?

$$X(p) \sim n_d \frac{h}{H} m_p v \frac{H^2}{D} \propto \frac{H}{D} \tag{498}$$

If we measure the fluxes of carbon and boron what we obtain will be the fraction H/D, which does not depend on the rate of injection. By measuring something which is large scale, we get access to the microphysics of the problem. We can also make this measurements energy dependent. **Therefore we can obtain the energy dependence of the diffusion coefficient.** We still have the limitation that we can only measure the combination H/D.
We measure that:

$$\frac{f_B}{f_C} \sim 0.3 \quad \text{at 10GeV per nucleon} \tag{499}$$

From this we have an estimate of the ratio $\frac{H}{D}$ in the galaxy. We can make some ansatz on the size H. Already in the '60s, it was pointed out that if we observe radio emission from our Galaxy, that should track the region where cosmic rays are. The reason is that radio emission is produced as the result of synchrotron radiation of electrons. If we look at the morphology of this emission (where it comes from), we see that the disk is extremely thin, but the radio emission is much more extended. That was the first evidence that the volume inside which cosmic rays are propagating is much bigger than the disk of the Galaxy, where we are. So we cannot say that outside the disk it is pretty empty space, because the halo contains magnetic fields that make the particles diffuse.
What are the orders of magnitude of our Galaxy sizes? The region where cosmic rays are living is of order of kpc (300pc is the size of the disk). If we use B/C to calculate H/D, then use the ansatz for H, we can find D and finally calculate the time cosmic rays spend in the Galaxy, which is given H^2/D. We find that approximately it is 100Myr at 10GeV (it is a decreasing function of energy).
What happens when we have an element with several isotopes? Some of them may be unstable. Nothing interesting would happen if the decay time of this unstable isotope was either much smaller or much larger than the 100 million years of cosmic rays confinement. We have to find some element with some isotope with several million years of decay time.
For example ^{10}Be and ^{26}Al have decay times of order of Myr, but ^{10}Be has a larger flux so it is more useful. It has three isotopes: ^{7}Be, ^{9}Be and ^{10}Be and from the point of view of spallation they are produced in the same amount if we are observing that in a laboratory.
In the cosmic rays, instead, we expect to see the abundances of berillium and boron to be roughly the same, since both of them are secondary nuclei produced through spallation. Berillium comes in threee isotopes and one of them is unstable. So what we see is a really tiny portion of ^{10}Be, compared to the laboratory results. The third pick is very low because ^{10}Be has had enough time to decay. This gives us a timescale it takes cosmic rays to leave the galaxy.
Notice the difference with respect to the boron/carbon case. In this case boron is stable and so everything depends on the ratio $X(p)/X_{cr}$. Here the grammage enters because we are producing berillium, but the rate at which we lose the unstable isotope does not depend on the grammage, it depends only on a timescale. Here we have two timescales: the decay timescale of berillium and the escape timescale. Therefore from B/C we can estimate H/D, but from berillium we can get at the quantity H^2/D. If we are able to measure both we can break the degeneracy between H and D.

IV. Transport of unstable nuclei: the case of berillium, ^{10}Be

We have that the equation of transport for the unstable berillium nuclei is:

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = -\cancel{Q} + 2hn_d\sigma_{sp}vf\delta(z) - f_p 2hn_d\tilde{\sigma}_{sp}v\delta(z) + \frac{f}{\tau_d} \tag{500}$$

Where we have canceled Q since there is no other source except spallation. The second term is the **loss term**, which is the rate at which ^{10}Be is destroyed by means of spallation (spallation of ^{10}Be). The third term is the **production term** from the spallation of primary nucleus (hence the p subscript). Finally, the last term is the **decay term** and it is also the only term which does not depend on $\delta(z)$. The decay time is the rest frame time multiplied by the Lorentz factor since the nucleus is moving (to account for the time dilation).
Notice that source terms always come with a minus sign, whereas loss terms have a plus sign.

Let's solve the equation :

- If $z \neq 0$, we have:

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = \frac{f}{\tau_d} \tag{501}$$

  The solution will be of the form:

$$f = A\exp\left[-\beta z\right] + B\exp\left[\beta z\right] \tag{502}$$

$$\Longrightarrow \begin{aligned} f' &= -A\beta\exp\left[-\beta z\right] + B\beta\exp\left[\beta z\right] \\ f'' &= A\beta^2\exp\left[-\beta z\right] + B\beta^2\exp\left[\beta z\right] \end{aligned} \tag{503}$$

  Remember that we are assuming that the diffusion coefficient is spatially constant. So we can write:

$$\begin{aligned} DA\beta^2\exp\left[-\beta z\right] + D\beta^2 B\exp\left[\beta z\right] &= \frac{A}{\tau_d}\exp\left[-\beta z\right] + \frac{B}{\tau_d}\exp\left[\beta z\right] \\ D\cancel{A}\beta^2 &= \frac{\cancel{A}}{\tau_d} \\ \beta^2 &= \frac{1}{D\tau_d} \\ \beta^{-1} &= \sqrt{D\tau_d} \end{aligned} \tag{504}$$

  Note that β^{-1} is the length scale ^{10}Be covers while diffusing in a decay time, bearing in mind that both D and τ_d are increasing functions of energy. β is the term entering the exponential, so it is saying the density must drop on a scale of the order of the diffusion length of the particles. But we have always a positive exponential, which comes from the free escape boundary condition at the edges of the halo. We are imposing them in what follows.
- For $z = 0$ we have that

$$f = f_0 \qquad \Longrightarrow \qquad A + B = f_0 \tag{505}$$

- For $z = H$ we have

$$f = 0 \qquad \Longrightarrow \qquad A\exp\left[-\beta z\right] + B\exp\left[\beta z\right] = 0 \tag{506}$$

  If we rename

$$y \equiv \exp\left[\frac{H}{\sqrt{D\tau_d}}\right] \tag{507}$$

  we can rewrite the previous expression as

$$\frac{A}{y} + By = 0 \tag{508}$$

We now have to solve for A and B:

$$\begin{cases} A + B = f_0 \\ \frac{A}{y} + By = 0 \end{cases} \Longrightarrow \begin{cases} A = -f_0\frac{y^2}{1-y^2} \\ B = \frac{f_0}{1-y^2} \end{cases} \tag{509}$$

Therefore:

$$f = -f_0\frac{y^2}{1-y^2}\exp\left[-\beta z\right] + \frac{f_0}{1-y^2}\exp\left[\beta z\right] \tag{510}$$

We are now left with the integration around the disk only

$$2D\frac{\partial f}{\partial z}\bigg|_{0^+} = 2hn_d\sigma_{sp}vf_0 - 2hn_d\tilde{\sigma}_{sp}vf_p \tag{511}$$

where

$$\frac{\partial f}{\partial z}\bigg|_{0^+} = f_0\beta\frac{y^2}{1-y^2} + \frac{f_0\beta}{1-y^2} = \frac{f_0\beta}{1-y^2}(1+y^2) \tag{512}$$

We can substitute back in the previous expression and obtain:

$$\begin{aligned} \frac{2D}{\sqrt{D\tau_d}}f_0\frac{y^2+1}{1-y^2} &= 2n_dh\sigma_{sp}vf_0 - 2n_dh\tilde{\sigma}_{sp}vf_p \\ f_0\left[2hn_d\sigma_{sp}v - 2\sqrt{\frac{D}{\tau_d}}\frac{1+y^2}{1-y^2}\right] &= 2hn_d\tilde{\sigma}_{sp}vf_p \end{aligned} \tag{513}$$

Observe that f_0 is linear in f_p, the primary distribution.

- What happens when $\tau_d \to \infty$? This means that the secondary is stable and we must reproduce the same thing as B/C.

$$\begin{aligned} \frac{H}{\sqrt{D\tau_d}} \to 0 &\implies y \simeq 1 + \frac{H}{\sqrt{D\tau_d}} \\ 2\sqrt{\frac{D}{\tau_d}}\frac{2}{1-1-\frac{2H}{\sqrt{D\tau_d}}} &= -2\frac{D}{H} \\ \implies f_0\left[2hn_d\sigma_{sp}v + 2\frac{D}{H}\right] &= 2hn_d\tilde{\sigma}_{sp}vf_p \\ f_0 &= f_p\frac{\tilde{X}/X_{cr}}{1+\frac{X}{X_{cr}}} \end{aligned} \tag{514}$$

  We obtain the same as for $\frac{B}{C}$. What does it physically mean that τ_d goes to infinity? It means that the energies are high. When energies are high, ^{10}Be behaves as stable. What does it mean large energies? If we take B/C at 10GeV per nucleon (then B/C$\sim$ 0.3) and we assume that the slope of the diffusion coefficient with energy is 0.5, we can calculate H^2/D and compare if it is smaller or larger than two million years multiplied by the Lorentz factor. This typically happens at around 100GeV-TeV energy scales.
- **What happens when energies are smaller, i.e. $\tau_d \to 0$?**

$$\begin{aligned} 2\sqrt{\frac{D}{\tau_d}}\frac{1+y^2}{1-y^2} &\overset{y\to\infty}{\sim} -2\sqrt{\frac{D}{\tau_d}} \\ f_0\left[2hn_d\sigma_{sp}v + 2\sqrt{\frac{D}{\tau_d}}\right] &= 2hn_d\tilde{\sigma}_{sp}vf_p \\ f_0\left[2\frac{h}{H}n_d\sigma_{sp}v\frac{H^2}{D} + 2\sqrt{\frac{D}{\tau_d}}\frac{H}{D}\right] &= 2\frac{h}{H}n_d\tilde{\sigma}_{sp}vf_p\frac{H^2}{D} \\ f_0\left[\frac{X}{X_{cr}} + \sqrt{\frac{H^2}{D\tau_d}}\right] &= f_p\frac{\tilde{X}}{X_{cr}} \end{aligned} \tag{515}$$

  where $\tilde{X}$ is the grammage of the primary.

So we can measure, in theory at last, the ratio $\frac{H^2}{D}$, but it's extremely difficult to measure ^{10}Be. It is difficult because in order to separate ^{10}Be from ^{9}Be, we have to do magnetic spectroscopy, we have to measure the Larmor radius of the two, which is difficult.
What we have been measuring (AMS experiment) is the total flux ^{7}Be $+^{9}$Be $+^{10}$Be, i.e. the total flux of berillium, and the total flux of boron. So that we obtain the ratio $\frac{\text{Be}}{\text{B}}$. If ^{10}Be is missing we see that this ratio is not one. Of course this can be done if have a large statistics of event, which we do have.
^{10}Be decays into boron. So not only the numerator is decreased, but the denominator is increased by the same amount. Therefore we can derive H and D separately. We have found H to be more or less 6kpc, even though we cannot exclude a larger value.

v. Acceleration of leptons

In galaxies what is important is the acceleration of leptons, i.e. electrons, because they are radiatively efficient. So far we assumed that the energy losses were negligible. When the energy losses are important the maximum energy not only depends on the age of the system and acceleration time, but also on the energy losses timescales. For protons these would be the **Bethe-Heitler** pair production and the **photopion** pair production. Whether energy losses are important or not depend on the situation.
What are the main radiation loss processes for relativistic electrons?

- Inverse Compton scattering (ICS)
- Synchrotron radiation
- Bremsstrahlung, the least important because of the conditions around the SN remnant

For the ICS the photons are those from the CMB. Remember that the galaxy is permeated with two types of photons:

- CMB photons
- optical light from the stars (1eV of energy density per cm^3, as the CMB)

In the case of SN remnants, ICS and synchrotron have the same energy scales

$$\begin{cases} \left(\frac{dE}{dt}\right)_{sync} \propto U_B^2 E_e^2 \\ \left(\frac{dE}{dt}\right)_{ICS} \propto U_{ph}^2 E_e^2 \end{cases} \tag{516}$$

where $U_B^2 = \frac{B^2}{8\pi}$.
For the ICS process we have:

$$e + \gamma_{CMB,opt} \longrightarrow e + \gamma \tag{517}$$

The synchrotron and ICS emissions are occurring at the same rate for $B \sim 3\mu$G.

For a supernova remnant the situation is different. Close to the shock there is compression of all the quantities: also the perpendicular component of the magnetic field at the shock is increased by a factor 4. This means that the energy density of the magnetic field behind the shock is increased by a factor 16. Therefore, **in the downstream of the shock synchrotron emission is the main channel of energy loss**.
In the upstream B is much larger than in the ISM because of the particles creating themselves perturbations. Energy losses in a SN remnant are severe (the rate of energy loss is around 100 times faster than in the ISM).
We need to compare the acceleration time $\tau_{acc} \sim \frac{D}{v_{shock}^2}$ with the loss timescale:

$$\tau_{syn} = \frac{E}{\frac{dE}{dt}} \sim \frac{1}{E} \tag{518}$$

It is a decreasing function of particle energy. At high energies, energy is lost faster and faster and τ_{acc} is very small. At a shock, for electrons:

$$\frac{D(E)}{v_{shock}^2} = \tau_{syn} \quad \Longrightarrow \quad \frac{E^\delta}{v_{shock}^2} = \frac{1}{E} \tag{519}$$

The first is increasing and the second decreasing. At a certain point they would be equal and this gives the maximum energy. This happens at energies of several TeV for a supernova range of parameters.

It is still true though that the spectrum of electrons which accelerate at supernova shock, for a strong shock is $N(E) \sim E^{-2}$. From a supernova remnant we only get out electrons and protons. We saw how protons and nuclei are transported in the Galaxy. We want to do the same with electrons. We also said that protons do not suffer appreciable energy losses, still they undergo **inelastic proton scattering**, that is the production of neutral and charged pions. This is an important energy loss channel, because even though protons lose only a fraction of their energy, most of the γ rays we observe in the disk of our Galaxy are due to the decay of neutral pions produced in this process (these retain 0.1 of the energy of the parent proton). The disk of the galaxy is very bright because of $p + p \to \pi_0$. Then the charged pions can decay and produce **positrons**:

$$\begin{aligned} p_{CR} + p_{ISM} &\longrightarrow p + p + \pi^0 + \pi^+ + \pi^- \\ \pi^0 &\longrightarrow \gamma + \gamma \\ \pi^+ + \pi^- &\longrightarrow \mu + \nu_\mu \longrightarrow e^\pm + \nu_e + \nu_\mu \end{aligned} \tag{520}$$

The astrophysical sources will only produce e^- and protons, but the transport can produce some amount of antimatter thanks to hadronic interactions in the Galaxy. So we want to calculate what we expect for both electrons and for secondary electrons and positrons. This is important because we expect something and then we see something completely different. There is a strong anomaly. In fact, **we expect that the ratio e^+/e^- is decreasing with energy, we measure it and it is increasing.**

The situation is similar to spallation of carbon. e^+ and e^- are secondary products as boron, the difference is that they suffer energy losses. This process occurs provided the energy of the parent proton is above 100MeV. Let's start again with the transport equation and primary electrons (the source term is associated with sources, like a SN):

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = -Q + \frac{f}{\tau_e(E)} \tag{521}$$

This is the transport equation for electrons, but **what about the diffusion coefficient?** Is it the same or different from the one of protons? The diffusion coefficient depends on r_L which is the same for electrons and protons provided they have the same energy. At first order we can say that it is the same for electrons and protons.

The source term is the same as that of protons, what changes is $N_e(E)$. For protons it was a power law up to the maximum energy determined by the age of the system. For electrons it will still be a power law but with maximum energy determined by the energy losses.

$$Q(E) = \frac{N_e(E)\mathcal{R}}{\pi R_d^2}\delta(z) \tag{522}$$

Observe that the equation we obtained is pretty similar to the one of berillium. Integrating (521) between $z = 0^-$ and $z = 0^+$, we get (the loss term is continuous and so it is the same in 0^+ and in 0^-):

$$2D\frac{\partial f}{\partial z}\bigg|_{0^+} = -\frac{N_e(E)\mathcal{R}}{\pi R_d^2} \tag{523}$$

For $z \neq 0$ and $z > 0$:

$$D\frac{\partial^2 f}{\partial z^2} = \frac{f}{\tau_e} \tag{524}$$

where τ_e is the loss timescale. This is similar to the berillium case. Therefore we can write the general solution as

$$f(z,p) = A\exp\left[-\beta z\right] + B\exp\left[\beta z\right] \tag{525}$$

with

$$\beta^2 = \frac{1}{D\tau_e} \tag{526}$$

How does it scale with energy?

$$\frac{E}{E^\delta} \sim E^{1-\delta} \tag{527}$$

δ can be derived observationally from the boron over carbon ratio and it is around 0.5:

$$\delta \sim 0.5 \tag{528}$$

So we have that the β scale is growing with energy. To find A and B we need boundary conditions, which are as usual:

$$\begin{cases} f(z=0) = f_0 \\ f(z=\pm H) = 0 \end{cases} \implies \begin{cases} A + B = f_0 \\ A\exp\left[-\beta z\right] + B\exp\left[\beta z\right] = 0 \end{cases} \tag{529}$$

The solution is:

$$f = f_0 \left\{ \frac{y^2}{y^2-1} \exp\left[-\beta z\right] - \frac{1}{y^2-1}\exp\left[\beta z\right] \right\} \tag{530}$$

We now have to calculate:

$$D\frac{\partial f}{\partial z}\bigg|_{0^+} = f_0 D\beta \frac{1+y^2}{1-y^2} \tag{531}$$

What does this imply?

$$Df_0 \frac{y^2+1}{y^2-1}\frac{1}{\sqrt{D\tau_e}} = \frac{N_e(E)\mathcal{R}}{2\pi R_d^2} \tag{532}$$

where the 2 factor in the denominator on the RHS comes from symmetry.
Therefore the solution to our problem, f_0, in the disk of the galaxy can be written as:

$$f_0(E) = \frac{N_e(E)\mathcal{R}}{2\pi R_d^2}\sqrt{\frac{\tau_e}{D}}\frac{y^2-1}{y^2+1}, \qquad y = \exp\left[\frac{H}{\sqrt{D\tau_e}}\right] \tag{533}$$

Remember that $\frac{H^2}{D}$ is the confinement time. So we have to analyze the situations where the confinement time is larger or small than the loss time.

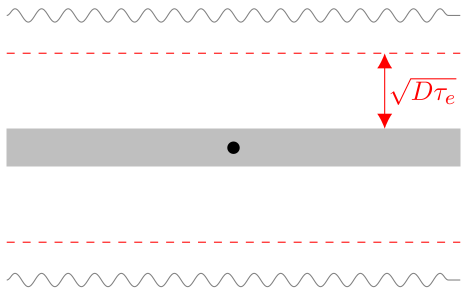
- When $\frac{H}{\sqrt{D\tau_e}} > 1 \implies \frac{H^2}{D} > \tau_e$, we have that the propagation time is longer than the loss time, so that the energy losses are important. In the limit $\frac{H}{\sqrt{D\tau_e}} \gg 1$, we have

$$\frac{y^2+1}{y^2-1} \to 1, \qquad f_0(E) = \frac{N_e(E)\mathcal{R}}{2\pi R_d^2}\sqrt{\frac{\tau_e}{D}} \tag{534}$$

  This means that, rewriting in more suitable way

$$f_0(E) = \frac{N_e(E)\mathcal{R}}{2\pi R_d^2}\frac{\tau_e(E)}{\sqrt{D\tau_e(E)}} \tag{535}$$

  the rate of injection is multiplied by the energy loss timescale, but now we do not divide by the whole volume, but only the one relevant for the losses, which means $\frac{H^2}{D} > \tau_e(E)$. So electrons are able to diffuse up to a certain height over the disk. All this reasoning holds if particles are only injected in the disk.



Note that this works if the particles are all injected in the disk, not when they can be produced elsewhere.

- When $\frac{H}{\sqrt{D\tau_e}} < 1$ the energy losses are not important. In the limit $\frac{H}{\sqrt{D\tau_e}} \ll 1$, we can Taylor expand $y \sim 1 + \frac{H}{\sqrt{D\tau_e}}$, therefore:

$$\frac{y^2-1}{y^2+1} = \frac{1+\frac{2H}{\sqrt{D\tau_e}}-1}{2+\frac{2H}{\sqrt{D\tau_e}}} \simeq \frac{H}{\sqrt{D\tau_e}} \tag{536}$$

So

$$f_0(E) = \frac{N_e(E)\mathcal{R}}{2\pi R_d^2}\sqrt{\frac{\tau_e}{D}}\frac{H}{\sqrt{D\tau_e}} = \frac{N_e(E)\mathcal{R}}{2\pi R_d^2 H}\frac{H^2}{D} \tag{537}$$

Observe that when energy losses are not important the solution is the same as for protons.

Rule of thumb

$$f_0 = \frac{\text{injection rate} \times \text{timescale}}{\text{effective volume}}$$

For electrons we have to care of the energy losses. Observe that at high energies, the energy loss timescale becomes really short compared to both the timescale of escaping the Galaxy and the timescale for diffusive motion from the source to us. In other words, we are always assuming that the sources are homogeneously distributed in the disk, which is not really the case. **The nature of sources is discrete, which is irrelevant when dealing with protons and nuclei. When is it important?**
Suppose d is the distance to the closest source, then when $\frac{d^2}{D(E)} > \tau_e$, we won't see particles from that source, even if it is not distant from us. Does this affect the flux of particles at the Earth?
The specific flux we get at Earth will be strongly affected if there were a SN in the neighborhood of our Solar System in the last million years. At low energies we don't care, because energy losses are not important and we see electrons coming also from very distant sources. At high energies, on the other hand, only local sources can possibly contribute to the flux.
How steep is the spectrum of electrons?

- When energy losses are not important the spectrum is the same as for protons:

$$f_0(E) \sim E^{-\gamma-\delta} \tag{538}$$

where $E^{-\gamma}$ is as usual the spectrum of the source.

- When energy losses are important, we have that $\tau_e < H^2/D$ and therefore

$$f_0(E) \sim E^{-\gamma-1-\delta/2+1/2} \sim E^{-\gamma-1/2-\delta/2} \tag{539}$$

In general, the spectrum of electrons is relatively steep. This implies that at high energies the flux becomes relatively small.
We need to see what happens to the so called secondary electrons and positrons.

v.1 Secondary e^- and e^+

The transport equation is as usual:

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] = \frac{f}{\tau_e} - q_e(E) \tag{540}$$

where f describes now secondary electrons and positrons. We have an injection term for secondary electrons. What is the injection term for secondary electrons? Remember that these electrons are the result of the *pp* scattering. The first simplifying assumption we can make is that the energy of the secondary is equal to a fraction of the energy of the parent proton:

$$E = \xi E_p, \quad \text{with} \quad \xi = \frac{1}{20} \tag{541}$$

Therefore the number of particles in the range dE will be given by:

$$\begin{aligned} q_e(E)dE &= n_p(E_p)dE_p n_d \sigma_{pp} c 2h\delta(z) \\ \implies q_e(E) &= n_p\left(\frac{E}{\xi}\right)\frac{1}{\xi}n_d\sigma_{pp}c2h\delta(z) \\ &= \frac{N\left(\frac{E}{\xi}\right)\mathcal{R}}{2\pi R_d^2 H}\frac{H^2}{D\left(\frac{E}{\xi}\right)}\frac{1}{\xi}n_d\sigma_{pp}c2h\delta(z) \end{aligned} \tag{542}$$

where we used c because we are assuming that the particles are relativistic and have written explicitly the number density of protons n_p (it is the same as f_0, we just renamed it), that we have calculated by solving the transport equation for protons.

If we look at the transport equation for secondary electrons (or positrons, since we are assuming they are produced in the same amount), it is the same as for the primary electrons provided that we make the following substitution:

$$\frac{N_e(E)\mathcal{R}}{2\pi R_d^2} \to \frac{N\left(\frac{E}{\xi}\right)\mathcal{R}}{2\pi R_d^2}\frac{H^2}{D\left(\frac{E}{\xi}\right)}\frac{1}{\xi}n_d\sigma_{pp}c\frac{h}{H} \tag{543}$$

The previous solution is the same as this new one, provided the substitution above. Let's do directly the two asymptotic cases:

- At low energies energy losses are not important, so:

$$f_{sec}(E) = \frac{N_p\left(\frac{E}{\xi}\right)\mathcal{R}}{2\pi R_d^2}\frac{H}{D\left(\frac{E}{\xi}\right)}\frac{1}{\xi}n_d h\sigma_{pp}c\frac{H}{D} \tag{544}$$

  In the same regime the spectrum of primary electrons is given by:

$$f_e(E) = \frac{N_e(E)\mathcal{R}}{2\pi R_d^2}\frac{H}{D(E)} \tag{545}$$

  So we can evaluate now the ratio[19]

$$\frac{f_{sec}(E)}{f_e(E)} = \frac{N_p\left(\frac{E}{\xi}\right)}{N_e(E)}\frac{1}{\xi}n_d\frac{h}{H}\sigma_{pp}c\frac{H^2}{D\left(\frac{E}{\xi}\right)} \tag{546}$$

[19] This is not the same as B/C, because there carbon was the parent of boron, while here electrons are not parents of secondary electrons.

How does this ratio scale with energy? The only part where energy enters is the diffusion coefficient:

$$\frac{f_{sec}(E)}{f_e(E)} \sim E^{-\delta} \tag{547}$$

So that f_{sec} decreases with energy, at least when energy losses are not important.

- When energy losses are important:

$$\begin{aligned} f_{sec}(E) &= \frac{N_p\left(\frac{E}{\xi}\right)\mathcal{R}}{2\pi R_d^2}\frac{H}{D\left(\frac{E}{\xi}\right)}\frac{1}{\xi}n_d h\sigma_{pp}c\frac{\tau_e}{\sqrt{D\tau_e}} \\ f_e(E) &= \frac{N_e(E)\mathcal{R}}{2\pi R_d^2}\frac{\tau_e}{\sqrt{D\tau_e}} \end{aligned} \tag{548}$$

Once again we find that:

$$\frac{f_{sec}(E)}{f_e(E)} \sim \frac{1}{D\left(\frac{E}{\xi}\right)} \sim E^{-\delta} \tag{549}$$

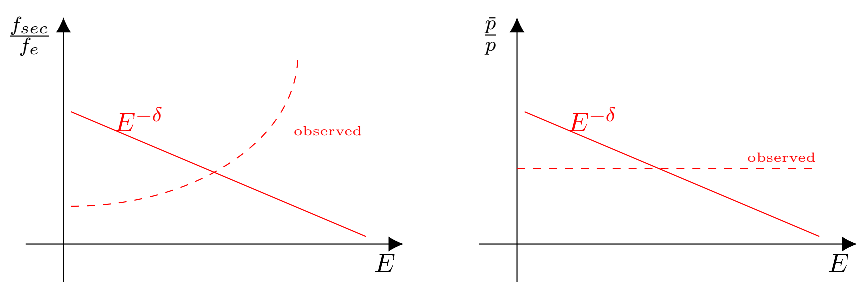
We have obtained the same ratio. **Whether we consider energy losses important or not, the ratio needs to be a decreasing function of energy. The problem is that measurements show an increase with energy**.
Where is the problem? Let's add a piece of information. We have analyzed the channel $p+p \to \pi + X$, but there is also the following channel:

$$p+p \to p+p+p+\bar{p}$$

So not only we produce positrons but also anti-protons. Therefore we can repeat the same calculations for anti-protons, because we do measure anti-protons in cosmic rays. Anti-protons behave similarly to protons (no severe energy losses). We can repeat the same calculations, taking into account the relatively high energy threshold of about 7GeV. We measure anti-protons. But what should we expect the spectrum of anti-protons to behave as a rule of thumb?
We produce anti-protons only in the disk (where we know the injection term). Once we know the rate of injection, we need to multiply by the escape time H^2/D and divide by the volume $2\pi R_d^2 H$. So also the ratio of secondary over primary needs to go as:

$$\frac{f_{\bar{p}}}{f_p} \sim \frac{1}{D} \sim E^{-\delta} \tag{550}$$

We measure it and it's flat. However, here we have some issues to be aware of. We have previously said that σ_{pp} is constant, but for anti-protons this cross section is a growing function of energy. Therefore, it will make the spectrum harder, $E^{-\delta+\epsilon}$. Unfortunately, little is known about the energy dependence of the cross section for anti-proton production, ϵ.



Coming back to the problem of positrons, we can think about another solution. We have to take into account that maybe there are sources of positrons in the Universe.

Pulsars actually produce positrons. The already cited *gedanken experiment* shows that a neutron star in vacuum is not a stable configuration.
If we have a magnet with its own dipolar magnetic field and we put it vacuum and we spin it, what happens is that the spinning magnetic field induces an electric field. The surface of the star is like a metal, where electrons are free to move. Generally the electric field is stronger than the gravitational force, so that the electrons try to leave the surface. The dipolar magnetic field of the neutron star is like $10^{13} - 10^{14}$gauss and the electrons leaving the surface and experiencing this strong magnetic field produce radiation. These photons are above the threshold for pair production, so they produce and electron and positrons pair. This process is repeated millions of times producing a large number of electrons and positrons. So as we anticipated, we cannot have a pulsar in vacuum, because even if we started from a vacuum configuration, immediately the surroundings of the pulsar will be filled with a sea of electrons and positrons, with a ratio of about a million pairs for each electron that is stripped of. We produce really a lot of pairs. The point is that we have an astrophysical source of electrons and positrons.
The contribution of pulsars is overwhelming compared to the secondary pairs. We don't have anti-protons in this process, which would require a much higher energy. We only produce electrons and positrons.
Also for these sources we have to consider their proximity to Earth and account for the possibility that we could not see electrons and positrons coming from them at sufficiently high energies.

Cosmic ray transport: some remarks

We have seen that the particle spectrum at accelerator follows the following momentum dependence:

$$f \sim p^{-\frac{3r}{r-1}}$$

where r is the compression factor at the shock. If we are inside the Galaxy, we do not observe this spectrum, but the diffusion coefficient D steps in and the spectrum gets modified by the dependence of D on momentum:

- D can be independent of p
- **Bohm diffusion**: $D(p) \propto p$
- **Kolmogorov diffusion**: $D(p) \propto p^{1/3}$

Cosmic rays diffusing in the Galaxy are subjected to:

- **Spallation** processes: it is a loss term for nuclei being spallated (as *carbon*) and an injection for nuclei that are the result of spallation process (as *boron*)

$$\tau_{sp} = \frac{1}{n\frac{h}{H}\sigma_{sp}c}$$

- **Decay** of unstable elements as the case of ^{10}Be
- **Energy losses**: these are important for leptons (ICS, synchrotron)